Open problem in the area of TSPPs (plane partitions).

I have previously published a paper on "conjugate" TSPPs

The method of this paper has been to define a kind of "basis" element for conjugate TSPPs. These are the shells which were defined in the paper. Any conjugate TSPP is then uniquely generated by a "sum" of these shells (as explained in that paper.

Before producing that paper, I had a more ambitious idea, namely could an arbitrary TSPP (not necessarily conjugate) be represented as a certain sum of shells? This in fact was the motivation in that paper for defining the more general idea of a TSPP n-shell which is given here: let us call a TSPP a 1-shell if it has any self-conjugate 1st row/column (as an ordinary partition) and all other entries are 1, e.g.:

1

2 1
1

3 2 1
2 1
1

4 4 2 2
4 1 1 1
2 1
2 1

The generating function for all 1-shell TSPPs is easily specified

We can naturally extend the concept of a 1-shell TSPP to that of a 2-shell TSPP (by embedding a smaller 1-shell TSPP in another 1-shell TSPP). And by an inductive process we may, if we wish, define an n-shell TSPP. Finally, by analogy with the concept of a Durfee square, we may define a Durfee "cube" for plane partitions as the largest cube which fits entirely inside the plane partition. The length of a side of this cube, n say, is easily seen to coincide with the largest n-shell contained entirely inside the given plane partition.

However the generating function for 2-shell TSPPs is not easy to obtain. I have not been able to solve how to carry out that step and it remains an open question. If it can be done it may (I believe) allow a generating function to be found for all TSPPs. Of course this has already been done but it might be simpler than the method employing the use of the theory of alternating sign matrices that was so eloquently described in Bressoud's book on the role of alternating sign matrices in this endeavour.

Moreover, the same type of generalisation may also work in obtaining a generating function for CSPPs. This, however, is another open question.

Reference:

A. Blecher, Geometry of totally symmetric plane partitions (TSPPs) with self-conjugate main diagonal,
Utilitas Math. 88 (2012), 223{235 (July volume).