Problem: Let $p := (2-\sqrt{3})/4 \approx 0.067$ and let ϕ satisfy

$$\phi(z) = \left((1-p)z + p\phi(\phi(z))\right)^2 . \tag{1}$$

Show that ϕ is analytic in a Camembert region about the singluarity at 1.

Note: for real $z \in (0,1)$, it is not hard to identify the behavior of ϕ as $z \to 1$, namely

$$1-\phi(z) \sim \frac{(1-z)}{4p} - \frac{\log(1/(4p)) + o(1)}{4p} \frac{1-z}{\log(1/(1-z))} . \tag{2}$$

This is done by tracking the iterates of ϕ^{-1} starting from any point y and showing that these approach 1 with difference

$$(4p)^n (cn + d(y) + o(1)) . \tag{3}$$

Any real x near 1 is $\phi^{-n}(y)$ for some $y \in (0,1/2)$. The fact that ϕ acts by replacing n by $n-1$ in (3) leads to (2).

If we could show Camembert convergence then singularity analysis would show that the coefficients of ϕ, call them a_n satisfy

$$a_n \sim \frac{c}{n^2 \log^2(n)}$$

with $c = \log(1/(4p))/(4p) \approx 4.915$. This would be interesting because ϕ is the generating fuction for the total progeny in a critical branching random walk.