# Algorithms and Pattern Avoiding Permutations

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May 27, 2013

# Permutations

A *permutation* is an arrangement of the integers $\{1, 2, \cdots, n\}$ in a line so that each integer appears exactly once.

## Example

Let $n = 7$, then 2317564, 3561742 and 2471653 are permutations.

# Pattern Containment

We say that a permutation $p_1 p_2 \cdots p_n$ contains the shorter permutation $q_1 q_2 \cdots q_k$ as a pattern if there is a subsequence of entries in $p$ that relate to each other as the entries of $q$. That is, $p$ contains $q$ as a pattern if there is a subsequence of $k$ entries $p_{i_1} p_{i_2} \cdots p_{i_k}$ so that $p_{i_a} < p_{i_b}$ if and only if $q_a < q_b$.

## Example

The permutation $p = 57821346$ contains the pattern $q = 132$ as shown in the figure.

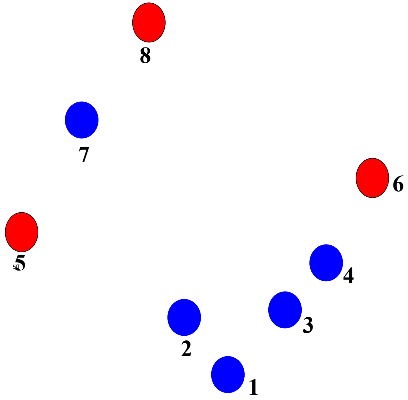



Figure: Containing the pattern 132.

# A Main Direction of Research

Let $S_n(q)$ be the number of permutations of length $n$ that avoid the pattern $q$. What can we say about $S_n(q)$? How large is $S_n(q)$?

**Conjecture:** (Stanley, Wilf, 1980): VERY SMALL!!!
Less than $c_q^n$, for some constant $c_q$ depending on $q$ only.
This conjecture was proved in 2003, by Marcus and Tardos.

# An Easy Special Case: Monotone Patterns

If $q$ is monotone, that is, if $q = 123 \cdots k$, then

$$S_n(q) < (k-1)^{2n}.$$

Indeed, let $p = p_1 p_2 \cdots p_n$ be a $q$-avoiding permutation. Let the *rank* $r(x)$ of each entry $x$ of $p$ be the length of the longest increasing subsequence ending at $x$.

**Example**

If $p = 68725431$, then the ranks of the $p_i$ are 1,2,2,1,2,2,2,1. The ranks of the entries $1, 2, \cdots$ are 1,1,2,2,2,1,2,2.

Then the two words

$$r(p_1)r(p_2)\cdots r(p_n)$$

and

$$r(1)r(2)\cdots r(n)$$

completely determine $p$ since entries of the same rank must be in decreasing order.
As each rank is at most $k-1$, the result follows.
It is much harder to prove, but true, that the constant $(k-1)^2$ is optimal for monotone patterns.

## Special cases for short $q$.

If $q$ is of length two, then we clearly have $S_n(12) = 1 = S_n(21)$.
If $q$ is of length three, then we clearly have

$$S_n(132) = S_n(231) = S_n(213) = S_n(312),$$

and also,

$$S_n(123) = S_n(321).$$

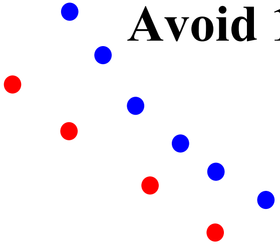
However, it is not trivial, but true, that $S_n(132) = S_n(123)$.

# The Simion-Schmidt Bijection, 1980

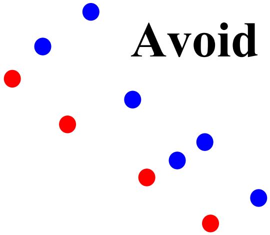
A *left-to-right minimum* in a permutation is an entry that is less than all entries on its left.

Classify $n$-permutations according the set and position of their left-to-right minima. Each such class contains exactly one 123-avoiding permutation and exactly one 132-avoiding permutation.

**Avoid 123**



**Avoid 132**



# Catalan numbers

So if $q$ is of length 3, then $S_n(q)$ does not depend on $q$. However, *what* is $S_n(q)$?
Let $a_n = S_n(132)$. It is then easy to see that the numbers $a_n$ satisfy the recurrence

$$a_n = \sum_{i=1}^{n} a_{i-1} a_{n-i}$$

with $a_0 = 0$, which is the famous Catalan recurrence.
So

$$a_n = S_n(132) = C_n = \frac{\binom{2n}{n}}{n+1} < 4^n.$$

# Patterns of Length 4

If $|q| = k = 4$, then a miracle happens. $S_n(q)$ will *depend* on $q$. Numerical evidence indicate that for $n \geq 7$,

$$S_n(1342) < S_n(1234) < S_n(1324).$$

In fact, these inequalities are known to be true.
Note that the monotone pattern is neither maximal nor minimal.

Any other pattern of length four is equivalent to one of the above three.

# The Pattern 1324

The pattern 1324 is *very difficult*. Even the exponential order of $S_n(1324)$ is not known. We will show two results, the oldest, and the most recent result, on this pattern.

### Theorem

*(Bóna, 1995) For all positive integers $n \geq 7$, the inequality $S_n(1234) < S_n(1324)$ holds.*

# Strong Classes

A *right-to-left maximum* in a permutation is an entry that is larger than all entries on its right.
Let us say that two permutations are in the same strong class if they have the same left-to-right minima and the same right-to-left maxima, and they are in the same positions.

## Example

Permutations 3612745 and 3416725 are in the same strong class.

It is easy to see that each non-empty class must contain exactly one 1234-avoiding permutation, since the entries that are neither left-to-right minima nor right to left maxima have to be in decreasing order.

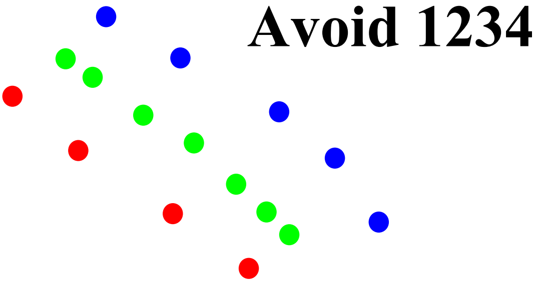
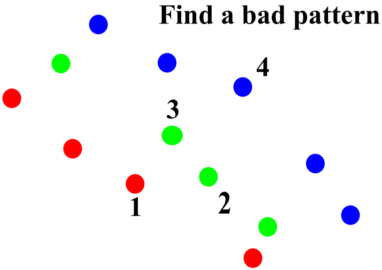
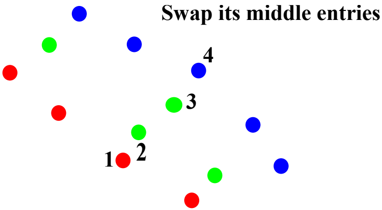



Figure: A 1234-avoiding permutation.

On the other hand, each non-empty strong class contains *at least one* permutation that avoids 1324. We prove this algorithmically.



Find a bad pattern, and swap its two middle entries. Repeat. It is easy to show that the algorithm does not change the strong class of the permutation.

When the algorithm stops, we have a 1324-avoiding permutation in our strong class.

How do we know that the algorithm *will* stop??

# An upper bound for 1324-avoiding permutations

Finding a good upper bound for the number $S_n(1324)$ of permutations of length $n$ that avoid 1324 is notoriously difficult. This is somewhat surprising, since for all other patterns of length four or less, the exact value of $S_n(q)$ is known.

We show the best current result, which improves a recent result of Claesson, Jelinek and Steingrímsson.

**Theorem**

*For all positive integers $n$, the inequality $S_n(1324) < (7 + 4\sqrt{3})^n$ holds.*

# Sketch of Proof

Let $p = p_1 p_2 \cdots p_n$ be a 1324-avoiding permutation, and let us color each entry of $p$ red or blue as we move from left to right, according the following rules.

1. If coloring $p_i$ red would create a 132-pattern with all red entries, then color $p_i$ blue, and
2. otherwise color $p_i$ red.

It can then be shown that the blue entries form a 213-avoiding permutation, while the red entries obviously form a 132-avoiding permutation.

There are at most $2^n$ possibilities for the set of the red entries, and there are at most $2^n$ possibilities for the positions in which red entries are placed.

Once the set and positions of the $k$ red entries are known, there are $C_k < 4^k$ possibilities for their permutation, just as there are $C_{n-k} < 4^{n-k}$ possibilities for the permutation of the blue entries, completing the proof of the inequality $S_n(1324) < 16^n$.

# A Better Estimate

Let us color each entry of the 1324-avoiding permutation $p = p_1 p_2 \cdots p_n$ red or blue as we have just seen.

Furthermore, let us mark each entry of $p$ with one of the letters $A$, $B$, $C$, or $D$ as follows.

1. Mark each red entry that is a left-to-right minium in the partial permutation of red entries by $A$,
2. mark each red entry that is not a left-to-right minimum in the partial permutation of red entries by $B$,
3. mark each blue entry that is not a right-to-left maximum in the partial permutation of blue entries by $C$, and
4. mark each blue entry that is a right-to-left maximum in the partial permutation of blue entries by $D$.

Call entries marked by the letter $X$ entries of *type* $X$.

Let $w(p)$ be the $n$-letter word over the alphabet $\{A, B, C, D\}$ defined above. In other words, the $i$th letter of $w(p)$ is the type of $p_i$ in $p$. Let $z(p)$ be the $n$-letter word over the alphabet $\{A, B, C, D\}$ whose $i$th letter is the type of the entry $i$ in $p$.

Example

Let $p = 3612745$. Then the subsequence of red entries of $p$ is 36127, the subsequence of blue entries of $p$ is 45, so $w(p) = ABABBCD$, while $z(p) = ABACDBB$.

Let us say that a word $w$ has a *CB-factor* if somewhere in $w$, a letter $C$ is immediately followed by a letter $B$.

**Lemma**

*If $p$ is 1324-avoiding, then $w(p)$ has no CB-factor.*

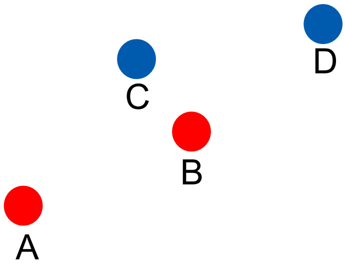



Figure: What a $CB$-factor would imply.

## Proof.

Assume that $C_1$ is the $i$th letter of $w(p)$, and $B_1$ is the $(i+1)$st letter of $w(p)$. That means that $p_i > p_{i+1}$, otherwise the fact that $p_i$ is blue would force $p_{i+1}$ to be blue. Since $p_i$ is not a right-to-left maximum, there is an entry $d$ on the right of $p_i$ (and on the right of $p_{i+1}$) so that $p_i < d$. Similarly, since $p_{i+1}$ is not a left-to-right minimum, there is an entry $a$ on its left so that $a < p_{i+1}$. Then $ap_ip_{i+1}d$ is a 1324-pattern, which is a contradiction. $\square$

In an analogous way, we can prove the following lemma.

**Lemma**

*If $p$ is 1324-avoiding, then there is no entry $i$ in $p$ so that $i$ is of type $C$ and $i+1$ is of type $B$. In other words, the word $z(p)$ obtained by rearranging $w(p)$ according to $p^{-1}$ has no CB-factors.*

So we can map the 1324-avoiding permutation $p$ into the pair $(w(p), z(p))$ or words over the alphabet $\{A, B, C, D\}$ that do not contain a $CB$-factor.

Crucially, this map is injective, so we can use the number of pairs of such words as an upper bound for $S_n(1324)$.

# Stack Sorting

We illustrate stack sorting by an example.

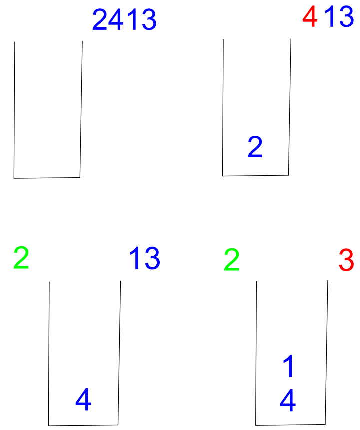
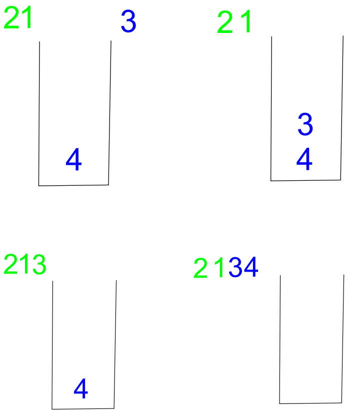



Figure: Entries in the stack have to increase top to bottom.




So $s(2413) = 2134 \neq 1234$, meaning that $p = 2413$ is not stack-sortable.

# Enumeration

### Theorem

*A permutation is stack sortable if and only if it avoids 231. So the number of stack sortable permutations of length n is the Catalan number* $c_n = \binom{2n}{n}/(n+1)$.

### Definition

A permutation $p$ is called two-stack sortable if $s(s(p)) = \mathrm{id}$. Similarly, a permutation is called $t$-stack sortable if $s^t(p) = \mathrm{id}$.

### Theorem

*(Zeilberger, 1995) If* $n \geq 1$, *then the number of two-stack sortable permutations of length n is*

$$W_2(n) = \frac{2\binom{3n}{n}}{(n+1)(2n+1)}.$$

# Why is this so difficult?

No formula, or even a good estimate, is known for the number of $t$-stack sortable permutations is known for the number $W_t(n)$ of $t$-stack sortable permutations if $t > 2$.

One reason for which these questions are difficult is that the class of $t$-stack sortable permutations is not monotone if $t > 1$. For instance, 35241 is 2-stack sortable, but 3241 is not. This means that the class of $t$-stack sortable permutations cannot be characterized in terms of (classical) pattern avoidance.

# Conjectures

Formulae

$$W_1(n) = \frac{\binom{2n}{n}}{n+1}$$

and

$$W_2(n) = \frac{2\binom{3n}{n}}{(n+1)(2n+1)}$$

suggest that maybe there exists a polynomial $p(n)$ so that

$$W_3(n) = \frac{\binom{4n}{n}}{p(n)}.$$

However, this is not true, because for $n = 11$, the number $W_3(n)$ has a huge prime factor.

# Right-greedy sorting

I still conjecture that

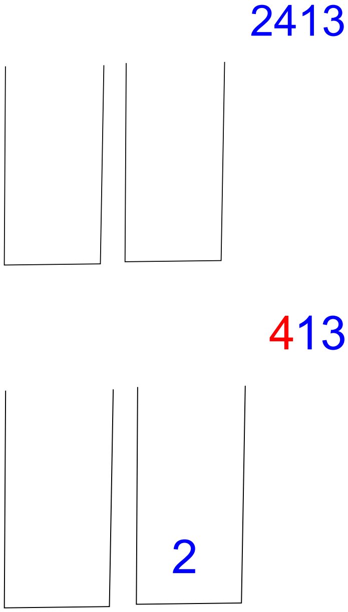
$$W_t(n) < \binom{(t+1)n}{n}.$$

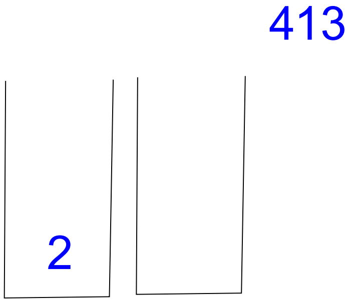
What is the rationale beyond this conjecture?

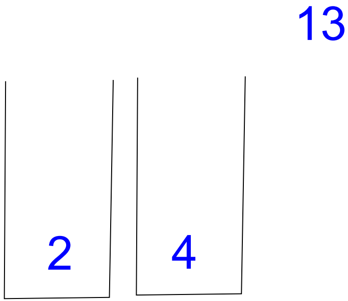
It can be proved that being $t$-stack sortable is equivalent to being sortable on $t$-stack in the *right-greedy* way.

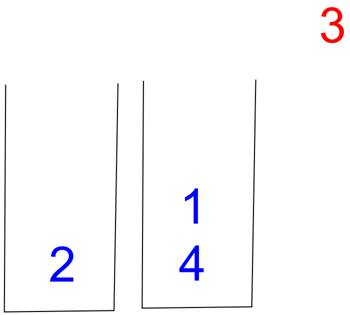
That is, we place $t$ stacks in a row, then place the input permutation on the right, and always make the *rightmost* move possible, where a move means moving an entry from a stack to the next stack on the left.

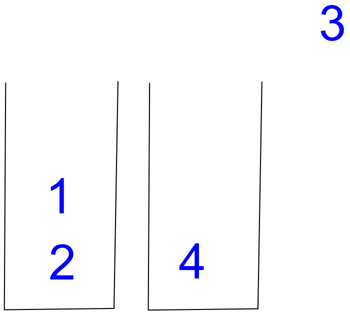
# The right-greedy sorting of 2413 on two stacks

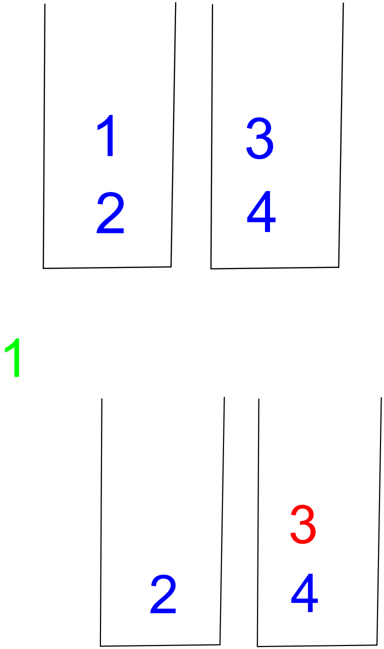



413
2


13
2
4


3
2
1
4


3
1
2
4


1
2
3
4
1
2
3
4

In this process, each of the $n$ entries moves $t+1$ times. If the permutation is $t$-stack sortable, it can be uniquely recovered from its movement sequence, that is, from a word of length $(t+1)n$ that contains each of $t+1$ letters $n$ times. What is missing is to show that the existing many constraints limit the number of acceptable words to $\binom{(t+1)n}{n}$.

Note that this is trivial if $t=1$, but not at all easy for $t=2$.

# Left-greedy sorting

Same as right-greedy sorting, but always make the leftmost possible move.

There is only one paper on this, by Atkinson, Murphy, and Ruskuc (2003).

Left-greedy sorting by two stacks is more efficient than right-greedy sorting by two stacks.

# A surprising coincidence

The numbers $a_n$ of such permutations of length $n$ have generating function

$$A(x) = \frac{32x}{-8x^2 + 20x + 1 - (1-8x)^{3/2}}.$$

However, it is known (B, 1996) that $A(x)$ is also the generating function of 1342-avoiding permutations!

This is interesting since permutations counted by $a_n$ are permutations that avoid all patterns of the infinite set

$$S = \{(2, 2m-1, 4, 1, 6, 3, 8, 5, \cdots, 2m, 2m-3) \mid m = 2, 3, 4...\}.$$

This is the only known example when a class of permutations avoiding one pattern (1342) is equinumerous to a class of permutations avoiding an infinite set of patterns.