Counting irreducible maps via substitution and bijections

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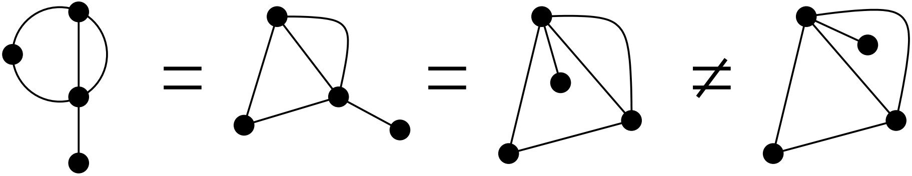
Introduction



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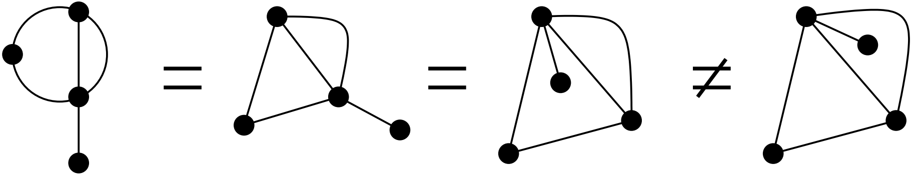

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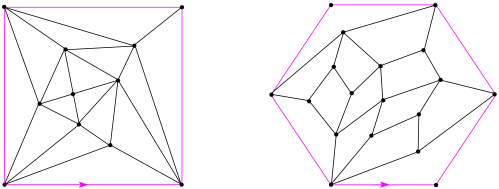

Irreducible triangular/quadrangular dissections

Introduction

Much of the recent progress in our understanding of maps (esp. their scaling limits) relies on the existence of bijections with trees. Many different bijections exist, and there is a case for providing a unified framework. Bernardi-Fusy ('11) and Albenque-Poulalhon ('13) introduced two such frameworks, both relying on a master bijection: (almost) every known bijection can be obtained as a restriction of one of them.

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As a novel application, Bernardi-Fusy obtained new enumerative results for maps with prescribed girth. Here, we explain how to recover (and extend) these results in two ways:

- the traditional way, using substitutions of generating functions,
- our own unified bijective framework: slice decomposition.

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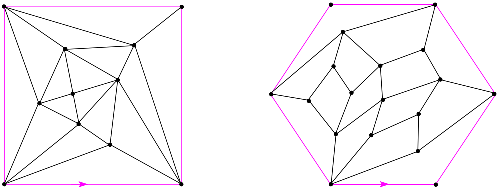
- the traditional way, using substitutions of generating functions,
- our own unified bijective framework: slice decomposition.

As a particular case, we recover the bijection between irreducible quadrangulations and binary trees, which has interesting applications to random generation of planar graphs [Fusy-Poulalhon-Schaeffer '08].

Definitions and notations

- Girth: smallest length of a cycle
- d-irreducible map ($d \geq 0$): a rooted map whose girth is at least d, and where every cycle of length d is the boundary of an inner face.
- $F_n^{(d)}(x_d; x_{d+1}, x_{d+2}, \ldots)$: multivariate generating function of d-irreducible maps with outer degree n, counted with a weight x_i per i-valent inner face ($i \geq d$).
- $F_n(x_1, x_2, \ldots)$: multivariate generating function of arbitrary maps ("well-known"), coincides with $F_n^{(0)}(0; x_1, x_2, \ldots)$.



Maps contributing to $F_4^{(3)}(x; 0, 0, \ldots)$ and $F_6^{(4)}(x; 0, 0, \ldots)$ respectively.

The generating function approach

Fundamental result

There exists d formal power series $X_1^{(d)}, X_2^{(d)}, \ldots, X_d^{(d)}$ in the variables $x_d, x_{d+1}, x_{d+2}, \ldots$ such that

$$F_n^{(d)}(x_d; x_{d+1}, x_{d+2}, \ldots) = F_n(X_1^{(d)}, X_2^{(d)}, \ldots, X_d^{(d)}, x_{d+1}, x_{d+2}, \ldots).$$

These series are determined in practice by the conditions

$$F_n^{(d)}(x_d; x_{d+1}, x_{d+2}, \ldots) = \begin{cases} \mathrm{Cat}(n/2) & \text{for } n < d \\ \mathrm{Cat}(d/2) + x_d & \text{for } n = d \end{cases}$$

which translate the fact that d-irreducible maps with outer degree at most d are either plane trees or made of a single d-valent face.

The generating function approach: proof idea

- Maps of girth at least d are $(d-1)$-irreducible maps without $(d-1)$-valent faces, hence counted by $F_n^{(d-1)}(0; x_d, x_{d+1}, \ldots)$.

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- Maps of girth d are alternatively obtained from d-irreducible maps by substituting each inner face of degree d with a rooted map of girth d and outer degree d (not reduced to a tree), thus

$$F_n^{(d-1)}(0; x_d, x_{d+1}, \ldots) = F_n^{(d)}(G_d(x_d, x_{d+1}, \ldots); x_{d+1}, \ldots)$$

where $G_d(x_d, x_{d+1}, \ldots) = F_d^{(d-1)}(0; x_d, x_{d+1}, \ldots) - \mathrm{Cat}(d/2)$.

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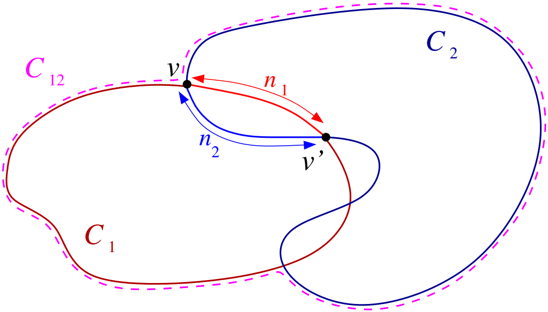
- There exists a power series X_d in $x_d, x_{d+1}, \ldots$ such that $G_d(X_d, x_{d+1}, \ldots) = x_d$, so that

$$F_n^{(d)}(x_d; x_{d+1}, \ldots) = F_n^{(d-1)}(0; X_d, x_{d+1}, \ldots)$$

  and our fundamental result follows by induction.

The generating function approach: proof idea

As a crucial ingredient in the proof of the substitution relation, we use the fact that in a rooted map of girth d, the maximal (outermost) cycles of length d do not overlap. This ensures that the substitution operation is bijective.



If C_1, C_2 are overlapping cycles of length d, we may find a cycle C_{12} of length d encircling them both (by the girth condition $n_1 + n_2 \geq d$, so that the length of C_{12} is at most, thus equal to, d).

The generating function approach: consequences

Simpler expressions are obtained in the bipartite case ($x_i = 0$ for i odd):

For $d = 2b$, the generating function of d-irreducible bipartite maps satisfies

$$\frac{\partial F_{2m}^{(d)}}{\partial x_d} = \binom{2m}{m-b} Y^{m-b}$$

where Y satisfies

$$x_d + \sum_{\ell=0}^{b} (-1)^{b-\ell} \binom{b+\ell}{2\ell} \mathrm{Cat}(\ell) Y^{b-\ell} + \sum_{j \geq b+1} \binom{2j-1}{j+b} x_{2j} Y^{b+j} = 0.$$

Similar (but more complicated) expression are obtained in the general case, and also without differentiating. Examples:

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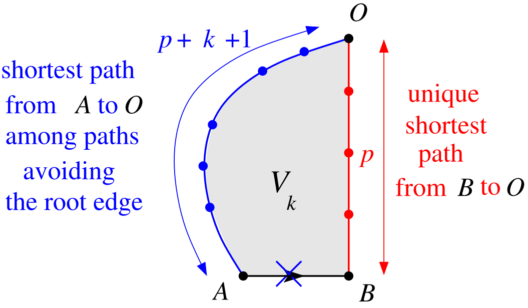
Similar (but more complicated) expression are obtained in the general case, and also without differentiating. Examples:

- $F_6^{(4)}(x;0,0,\ldots) = \sum_{n\geq 0} \frac{6}{n+2} \mathrm{Cat}(n) x^{n+2}$ [Mullin-Schellenberg, 1968]
- $F_8^{(6)}(x;0,0,\ldots) = 14 + 8\,x + 4\,x^2 + 8\,x^3 + 34\,x^4 + 192\,x^5 + 1264\,x^6 + \cdots$

The slice decomposition approach

It consists in interpreting combinatorially some quantities appearing in the previous approach. A k-slice is a rooted map with boundary constraints:

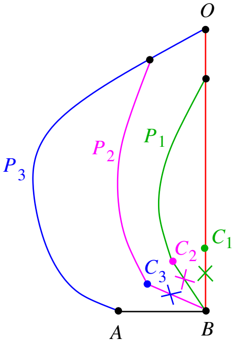


It is a slight generalization of the notion of slice introduced in [B.-Guitter 2010].

We denote by $V_k^{(d)}(x_d; x_{d+1}, \ldots)$ the generating function of d-irreducible k-slices (with p arbitrary).

The slice decomposition approach

Interestingly, d-irreducible k-slices admit a natural recursive decomposition, which easily translates into algebraic equations:



$$V_k^{(d)} = x_d \delta_{k,d-2} + \sum_{q \geq 1} \sum_{\substack{m_i \geq 1, \; i=1,\cdots,q \\ m_1+\cdots+m_q = k+2}} \prod_{i=1}^{q} V_{m_i}^{(d)}, \qquad -1 \leq k \leq d-2.$$

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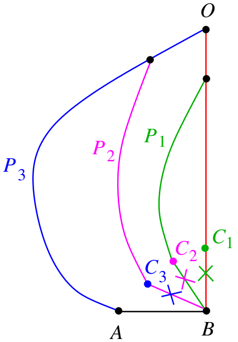
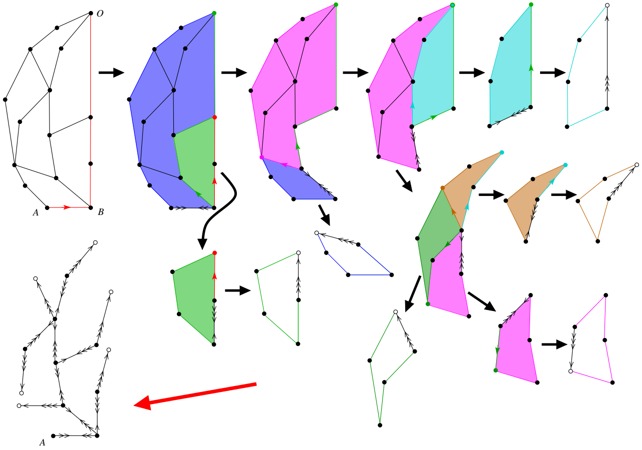


The slice generating functions $V_k^{(d)}$ are characterized as solutions of a system of algebraic equations. We recover the results of Bernardi-Fusy in the case of maps with prescribed girth.

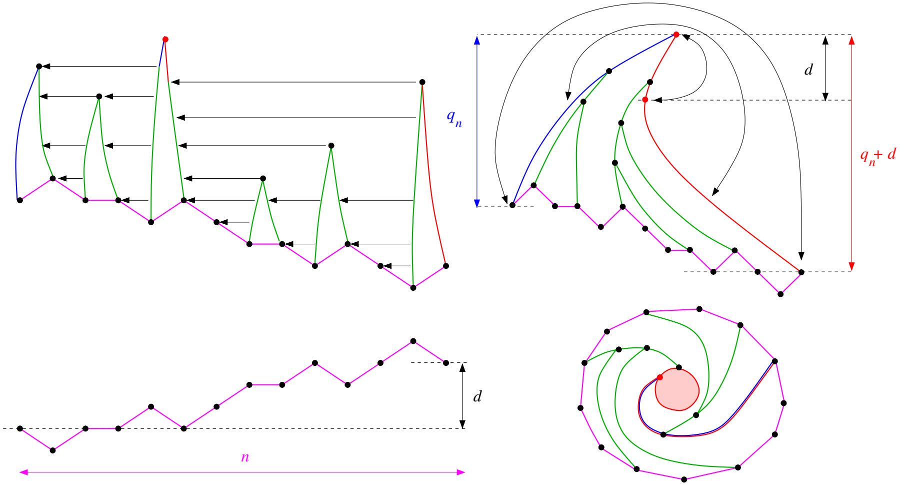
The slice decomposition approach

The recursive decomposition of d-irreducible k-slices yields a bijection with suitable trees.



The slice decomposition approach

Annular maps (counted by $\frac{\partial F_n^{(d)}}{\partial x_d}$) are obtained by gluing slices together, in particular this yields a bijective proof of the above bipartite formula.



Conclusion

Substitution and slice decomposition form two different (but related) approaches to the enumeration of irreducible maps.
Some further results:

- In the case of 3-irreducible triangulations and 4-irreducible quadrangulations, slices turn out to be related to "naturally embedded trees" [Bousquet-Mélou '05, Kuba '09].
- We obtain an expression for the generating function of irreducible maps with several boundaries, extending a formula in [Collet-Fusy '12].