Label patterns in mappings

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Ongoing work with Alois Panholzer

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Mappings and their graph representation

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Random mappings

Every one of the n^n many n-mappings occurs equally likely.

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Graph representation

M can also be represented by its *functional graph*:

- vertex set $[n]$
- directed edges $(i, M(i))$ for all $i \in [n]$
- outdegree of each vertex is exactly equal to 1

Mappings and their graph representation, Example

$$M = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 \\ 7 & 1 & 10 & 14 & 19 & 7 & 11 & 10 & 16 & 10 & 19 & 7 & 19 & 4 & 7 & 4 & 13 & 13 & 1 \end{pmatrix}$$

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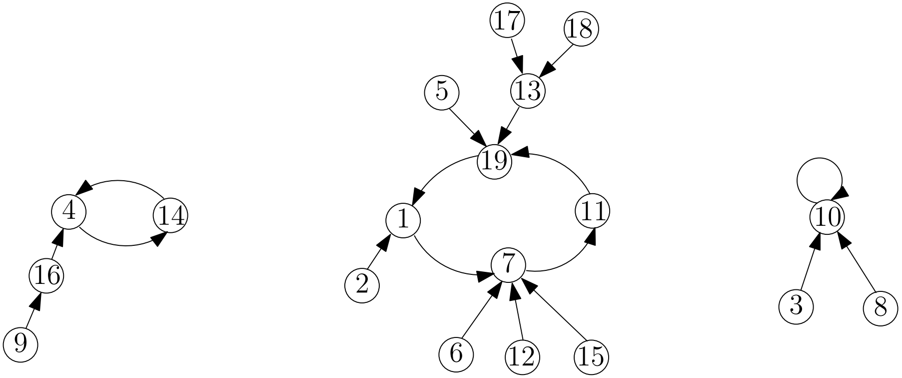

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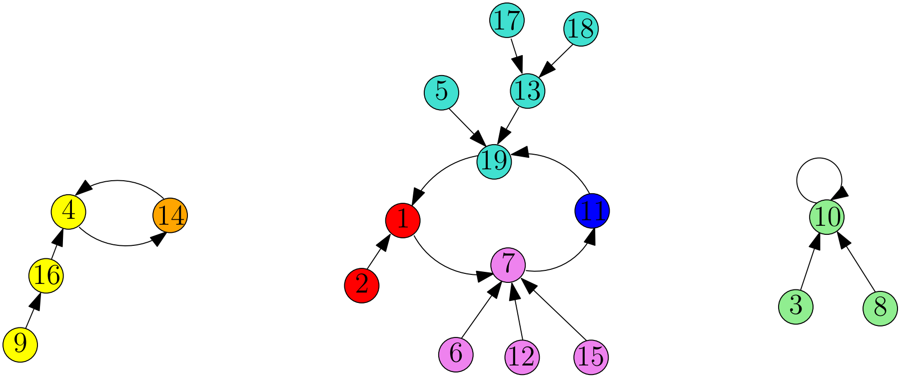

Structure of functional graphs

Every connected component of a functional graph is a collection of Cayley trees arranged in a cycle.

In the language of the symbolic method:

$$\mathcal{M} = \textsc{Set}(\mathcal{C}) \qquad M(z) = e^{C(z)}$$

$$\mathcal{C} = \textsc{Cyc}(\mathcal{T}) \qquad C(z) = \log\left(\frac{1}{1-T(z)}\right)$$

$$\mathcal{T} = \mathcal{Z} \star \textsc{Set}(\mathcal{T}) \qquad T(z) = z \cdot e^{T(z)}$$

What is known about mappings?

Flajolet and Odlyzko. *Random mapping statistics*. Advances in cryptology - EUROCRYPT'89. Springer Berlin Heidelberg, 1990.

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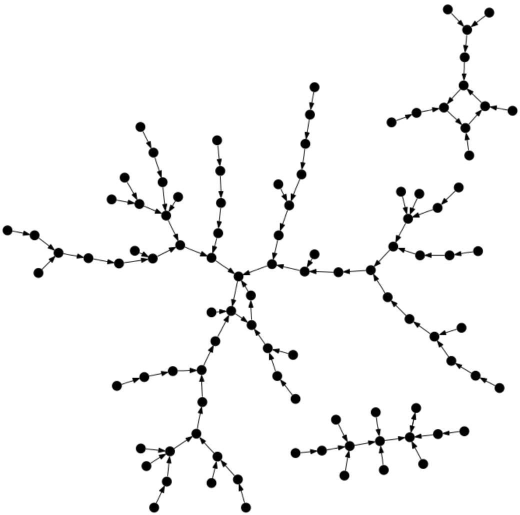
Expectations of some parameters:

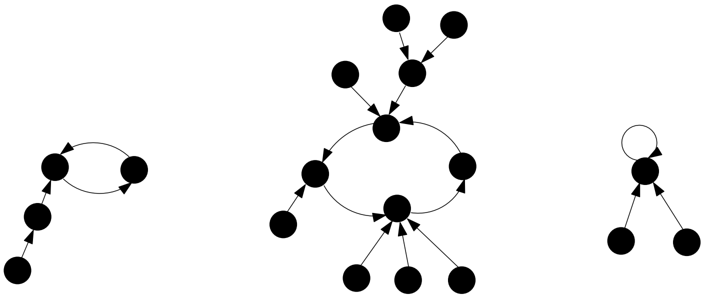
- # components $\sim \frac{1}{2}\log(n)$
- # cyclic nodes $\sim \sqrt{\pi n/2}$
- # terminal nodes $\sim e^{-1}n$
- size of largest tree $\sim c \cdot n$ with $c \approx 0.48$
- size of largest component $\sim d \cdot n$ with $c \approx 0.76$

What does a random mapping "look" like?



A random mapping of size 100, borrowed from *Analytic Combinatorics*.

What is our approach?



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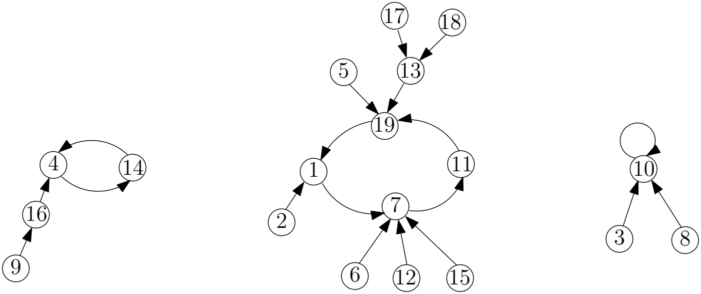

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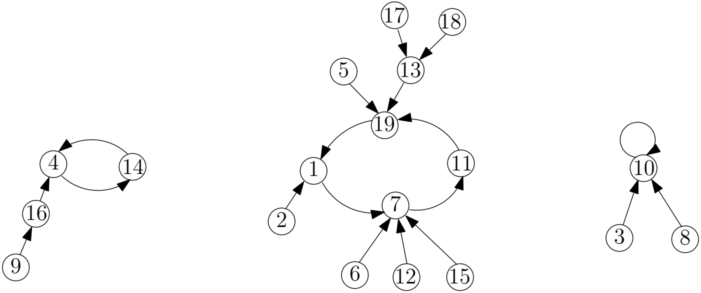

Instead of considering structural parameters, we consider *label parameters*. More specifically, we are interest in occurrences of *label patterns*.

Starting points

- ascents and descents in mappings: easy
- increasing and decreasing mappings: easy
- *alternating mappings*: studied by Alois Panholzer in *Alternating mapping functions*, 2012, submitted.

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Definition and Example

Ascending run

Maximal ascending sequence:

$$i < M(i) < M^2(i) < \ldots < M^k(i)$$

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Other characterization

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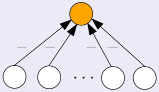

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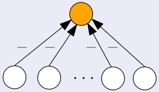

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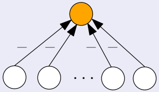

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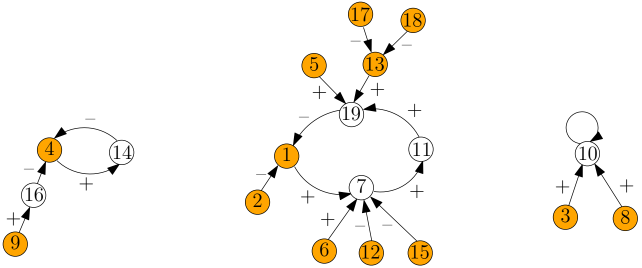

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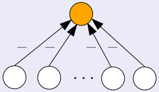

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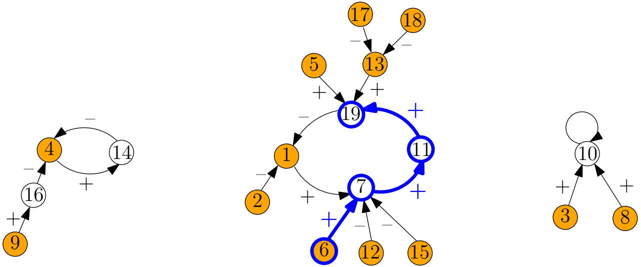

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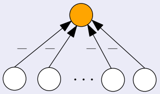

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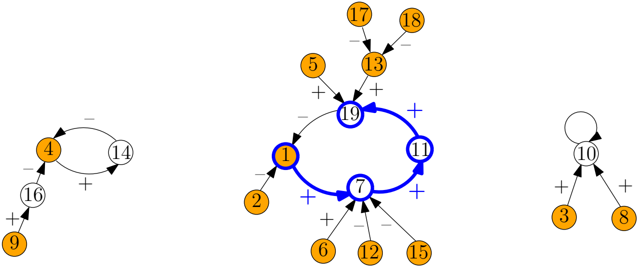

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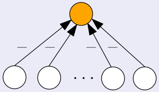

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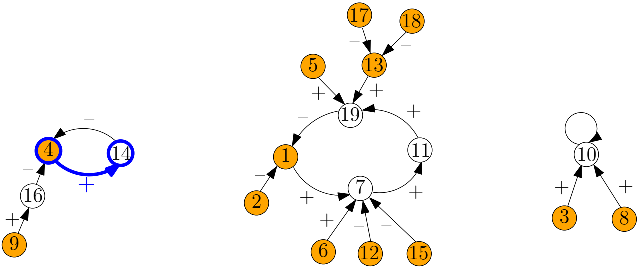

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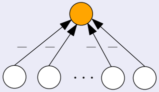

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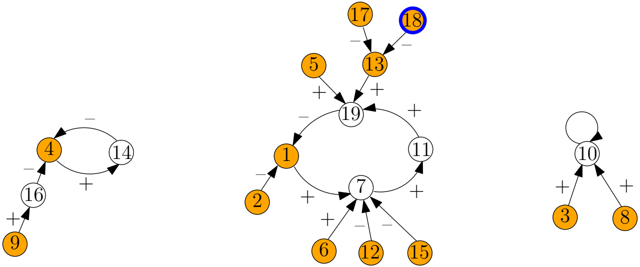
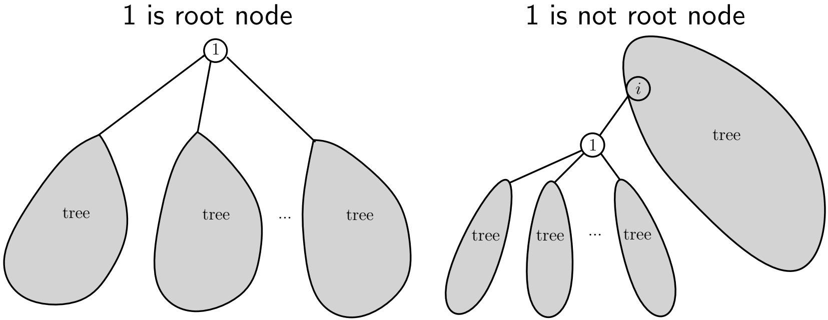
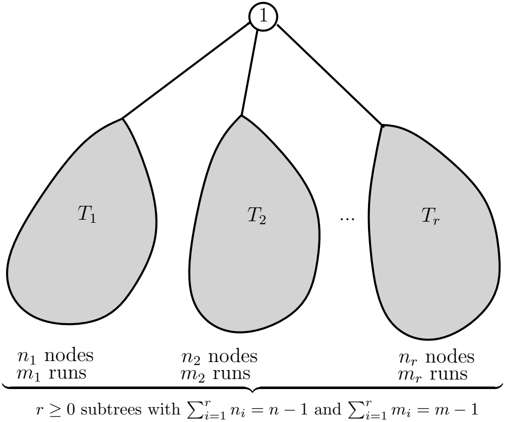


Runs in trees: decomposition idea

Decomposition with respect to the node 1 - two different cases to consider:



Runs in trees: Case 1



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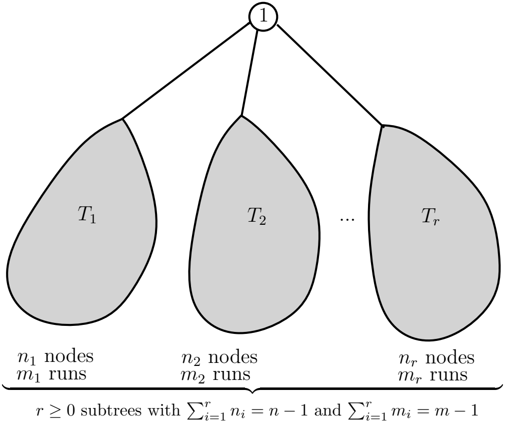
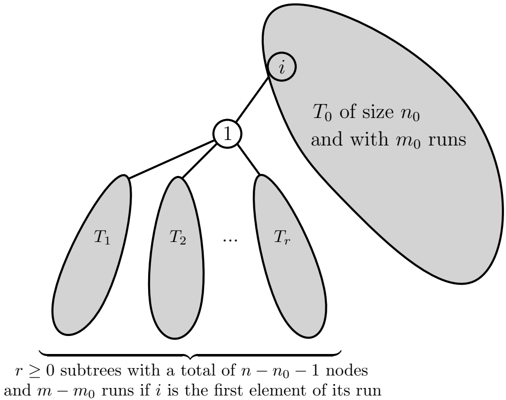


$$\sum_{r\geq 1} \frac{1}{r!} \sum_{\substack{\sum n_i = n-1 \\ \sum m_i = m-1}} \binom{n-1}{n_1, \ldots, n_r} T_{n_1,m_1} \cdots T_{n_r,m_r}$$

Runs in trees: Case 2a



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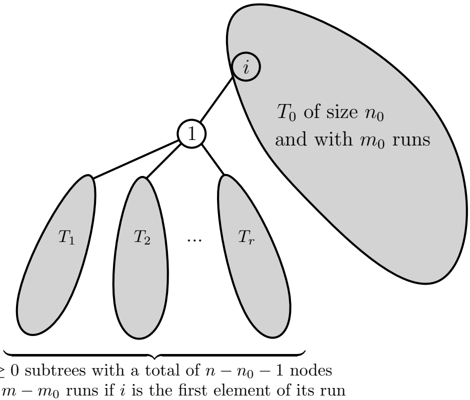
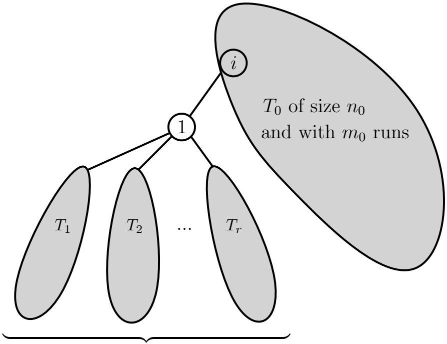


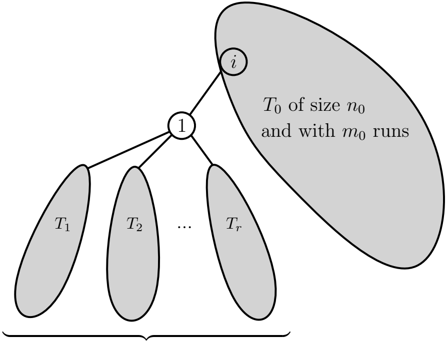
$$\sum_{\substack{1 \leq n_0 \leq n-1 \\ 1 \leq m_0 \leq m}} \sum_{r \geq 0} \frac{1}{r!} \sum_{\substack{\sum n_i = n - n_0 - 1 \\ \sum m_i = m - m_0}} m_0 \cdot \binom{n-1}{n_0, n_1, \ldots, n_r} T_{n_0,m_0} \cdot T_{n_1,m_1} \cdots T_{n_r,m_r}$$

Runs in trees: Case 2b



$r \geq 0$ subtrees with a total of $n - n_0 - 1$ nodes
and $m - m_0 - 1$ runs if i is not the first element of its run

Runs in trees: Case 2b



$$\sum_{\substack{1 \leq n_0 \leq n-1 \\ 1 \leq m_0 \leq m}} \sum_{r \geq 0} \frac{1}{r!} \sum_{\substack{\sum n_i = n - n_0 - 1 \\ \sum m_i = m - m_0 - 1}} (n_0 - m_0) \cdot \binom{n-1}{n_0, n_1, \ldots, n_r} T_{n_0,m_0} \cdots T_{n_1,m_1} \cdots T_{n_r,m_r}$$

Runs in trees: generating function and exact formula

With $T(z,v) = \sum_{n\geq 1}\sum_{m\geq 0} T_{n,m}\frac{z^n}{n!}v^m$, putting all together leads to:

$$(1 - vze^{T(z,v)})T_z(z,v) - v(1-v)e^{T(z,v)}T_v(z,v) - ve^{T(z,v)} = 0.$$

Solving this PDE and adapting to the initial conditions gives:

$$z \cdot e^T = \ln\Big((e^T - 1 + v)/v\Big).$$

Theorem: number of trees of size n with exactly m runs

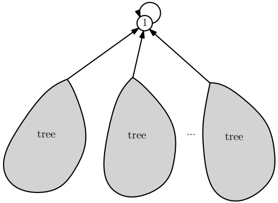
$$\begin{aligned} T_{n,m} &= \binom{n-1}{m-1}\sum_{\ell=0}^{m-1}(\ell+1)^{n-1}(-1)^{m-1-\ell}\binom{m-1}{\ell} \\ &= (n-1)^{\underline{m-1}} \cdot \left\{ {n \atop m} \right\}, \end{aligned}$$

where $\left\{ {n \atop m} \right\}$ are the Stirling numbers of the second kind.

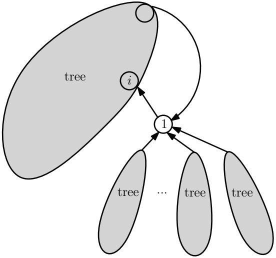
From trees to connected mappings: decomposition idea

Decomposition with respect to node 1 - three different cases to consider:

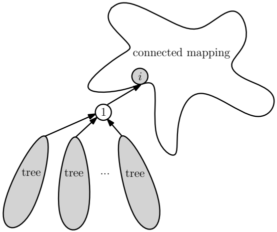
1 is root node
in cycle of length 1



1 is root node
in cycle of length ≥ 2



1 is not
cyclic node



Countings runs in mappings: Generating function

With $C(z,v) = \sum_{n\geq 1}\sum_{m\geq 0} C_{n,m}\frac{z^n}{n!}v^m$ this decomposition leads to

$$C_z(z,v) = vC_v(z,v)e^{T(z,v)} + v(zC_z(z,v) - vC_v(z,v))e^{T(z,v)} + T_z(z,v).$$

This PDE has the following solution:

$$C(z,v) = \ln\left(\frac{ve^{H(z,v)} + 1 - v}{ve^{H(z,v)}\big(1 - H(z,v)\big) + 1 - v}\right),$$

with with $H(z,v)$ defined by

$$z = \frac{H}{ve^H + 1 - v}.$$

With $M(z,v) = \sum_{n\geq 0}\sum_{m\geq 0} M_{n,m}\frac{z^n}{n!}v^m$ we then have

$$M(z,v) = e^{C(z,v)} = \frac{ve^{H(z,v)} + 1 - v}{ve^{H(z,v)}\big(1 - H(z,v)\big) + 1 - v}.$$

Counting runs in mappings

Theorem

Enumeration formula

The number $M_{n,m}$ of n-mappings with exactly m runs is given by:

$$M_{n,m} = \frac{n!}{(n-m)!} \left\{ n \atop m \right\}.$$

Expected number of runs

For the number R_n of runs in a random n-mapping, we have:

$$\mathbb{E}(R_n) \sim (1 - e^{-1})n$$

An interesting connection

Theorem

For all $n, m \geq 1$ *it holds that:*

$$M_{n,m} = n \cdot T_{n,m}$$

An interesting connection

Theorem

For all $n, m \geq 1$ it holds that:

$$M_{n,m} = n \cdot T_{n,m}$$

Also: $M_n = n^n = n \cdot T_n$.

Can $M_{n,m} = n \cdot T_{n,m}$ be proven bijectively?

The bijection of Kreweras and Moszkowski, Definition

Bijection $m : \mathcal{T}_n \longrightarrow [n]^{n-1}$, $T \mapsto m(T) := m_T$.
Define the following sets of nodes:

$$\mathcal{S} := \left\{ i \in [n-1] : \exists k \text{ s.t. } T^k(i) > i \right\}$$

and

$$\mathcal{S}' = [n] \setminus \mathcal{S} = \{x_1 < x_2 < \ldots < x_k = n\}.$$

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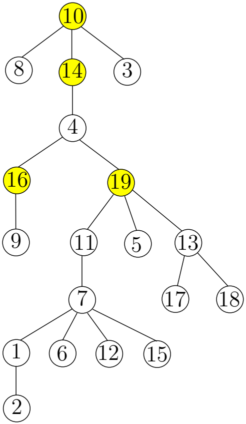
$$\mathcal{S}' = [n] \setminus \mathcal{S} = \{x_1 < x_2 < \ldots < x_k = n\} .$$

The sequence $m_T = m_T(1), \ldots, m_T(n-1)$ associated to T is defined as follows:

$$\text{If } i \in \mathcal{S}, \text{ then } m_T(i) = T(i),$$
$$\text{For } 1 \leq j \leq k-1, m_T(x_j) = T(x_{j+1}).$$

The bijection of Kreweras and Moszkowski, Example



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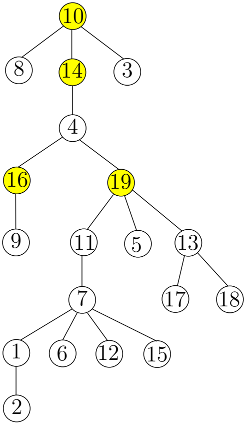

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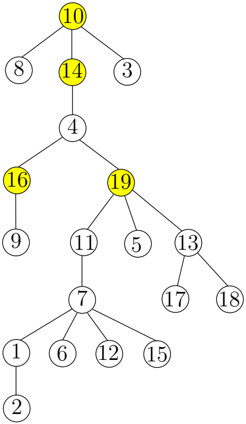

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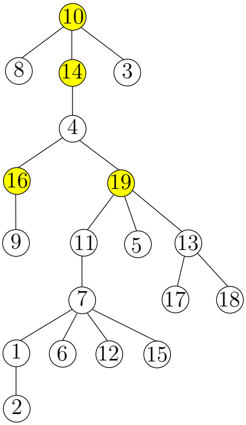

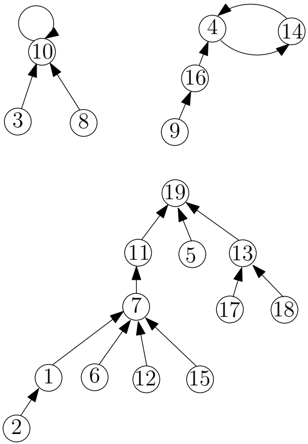

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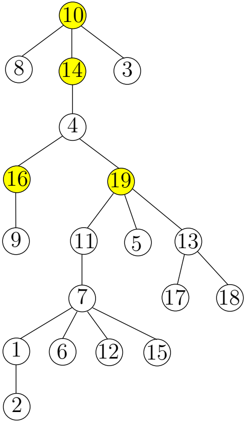

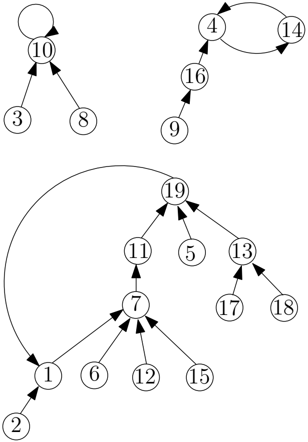

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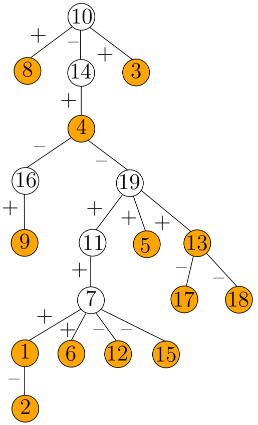

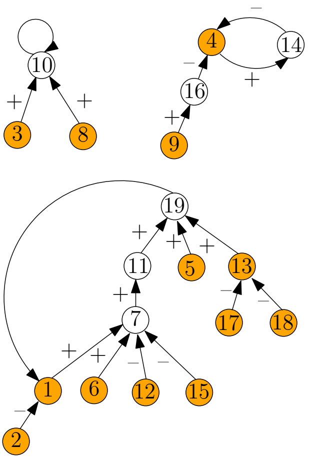

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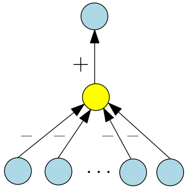
All neighbours have larger labels.



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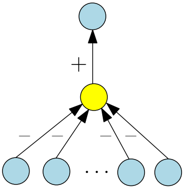
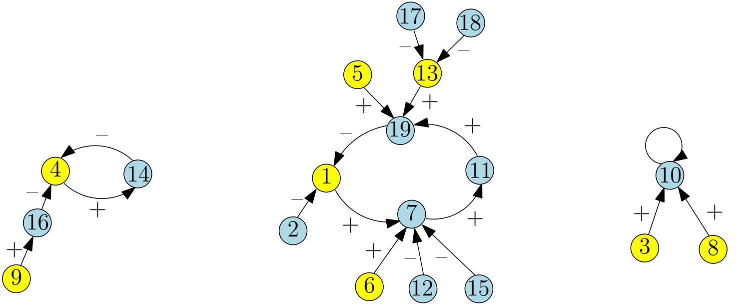




Local minima in trees

Theorem (Alois Panholzer and Georg Seitz, 2011)

The generating function $\tilde{T}(z,v) = \sum_{n\geq 1}\sum_{m\geq 0} \tilde{T}_{n,m}\frac{z^n}{n!}v^m$ of local minima in Cayley trees is characterized by the following functional equation:

$$\tilde{T}(z,v) = z \cdot \left(e^{T(z,v)} + v - 1\right)$$

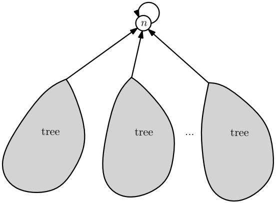
and thus the number of trees of size n with exactly m local minima is:

$$\tilde{T}_{n,m} = \frac{n!}{m!}\left\{ {n-1 \atop n-m} \right\}$$

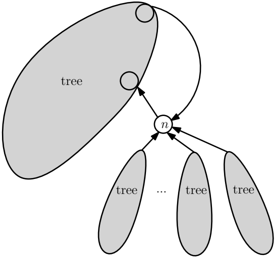
Local minima in mappings: Decomposition idea

Decomposition with respect to node n - three different cases to consider:

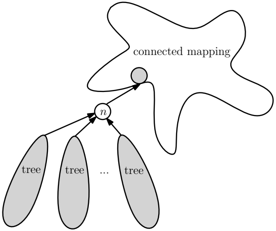
n is root node
in cycle of length 1



n is root node
in cycle of length ≥ 2



n is not
cyclic node



Counting local minima in mappings

Theorem

Enumeration formula

the number $\tilde{M}_{n,m}$ of n-mappings with exactly m local minima is given by:

$$\tilde{M}_{n,m} = \frac{(n-1)!}{(m-1)!} \sum_{k=0}^{n} \binom{n}{k} \left\{ {k \atop n-m} \right\}.$$

Expected number of local minima

For the number L_n of local minima in a random n-mapping, we have:

$$\mathbb{E}(L_n) \sim e^{-1} n$$

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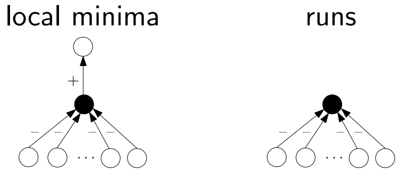
What have we done so far? What is new in what we have done?

- label parameters and not structural ones
- decomposition of trees/mappings with respect to smallest/largest node and not root node

What is to be done next?

- We have studied “tree patterns” in mappings:



- What about avoiding such tree patterns? Avoiding other label patterns such as 123?

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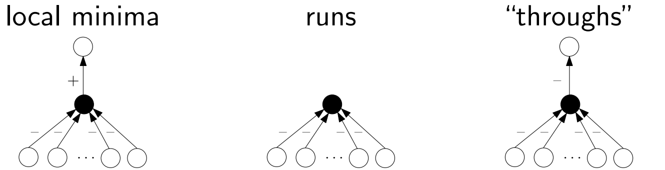

- What about avoiding such tree patterns? Avoiding other label patterns such as 123?

Future work

- Other statistics that are defined for permutations
  - inversions: have been studied for trees (Alois Panholzer and Georg Seitz, 2011), leads to Airy distribution
  - left-to-right-minima
  - number of *alternating runs*
  - ?
- In trees the statistics "number of local minima" and "number of leaves" are equidistributed. This is not the case for mappings (but "nearly"). Are there other "label parameters" that have connections to "structural parameters"?