Coefficients of Positive Algebraic Functions

Michael Drmota

joint work with Cyril Banderier

Institute of Discrete Mathematics and Geometry

Vienna University of Technology

A 1040 Wien, Austria

michael.drmota@tuwien.ac.at

www.dmg.tuwien.ac.at/drmota/

AofA 2013, Menorca, May 31, 2013

$\mathbb{R}_+$-**Algebraic functions**

System of polynomial functional equations

P_j ... polynomials with **non-negative** coefficients

$$\begin{cases} y_1 = P_1(z, y_1, \ldots, y_d) \\ \vdots \\ y_d = P_d(z, y_1, \ldots, y_d) \end{cases}$$

... **well defined** system of equations (see Bruno's talk):
there is a unique set of power series solutions

$$\boxed{y_1 = a_1(z), \ldots, y_d = a_d(z)}$$

with non-negative coefficients $a_{j;n} = [z^n]\, a_j(z)$.

Remark. $a_j(z)$ are algebraic functions.

$\mathbb{R}_+$-Algebraic functions

Example

$$
\begin{aligned}
y_1 &= zy_1 + z^2 y_2 y_3^2 + y_4^3, \\
y_2 &= z + z y_3^5 y_4, \\
y_3 &= z^2 + z y_1 y_3^3 + y_2^2, \\
y_4 &= z y_2 y_3 + z^3 y_4
\end{aligned}
$$

$\mathbb{R}_+$-Algebraic functions

Applications

- Context free grammars
- Combinatorial enumeration problems
- Species
- Random generation
- Automatic sequences
- Tiling problems
- ...

$\mathbb{R}_+$-**Algebraic functions**

Problem

Given a (well defined) $\mathbb{R}_+$-algebraic system of equations $\mathbf{y} = \mathbf{P}(z, \mathbf{y})$ with solutions $y_j = a_j(z)$.

What is the **asymptotic behavior** of the coefficients

$$\boxed{\boxed{a_{j;n} = [z^n]\, a_j(z)}} \quad ???$$

Binary Trees

Generating Function. $b(z) = \sum_{n \geq 0} b_n z^n$

$$\boxed{b(z) = 1 + z\, b(z)^2}$$

$$b(z) = \frac{1 - \sqrt{1-4z}}{2z} = 2 - 2\sqrt{1-4z} + \cdots$$

$$b_n = [z^n] b(z) = \frac{1}{n}\binom{2n}{n} \sim \frac{4^n}{\sqrt{\pi}\, n^{3/2}}$$

A Single Functional Equation

Theorem (Bender, Canfield, Meir & Moon)

Suppose that $a(z)$ satisfies $\boxed{a(z) = \Phi(z, a(z))}$, where $\Phi(z, y)$ has a power series expansion at $(0, 0)$ with non-negative coefficients, $\Phi(0, 0) = 0$, $\Phi_{yy}(z, y) \neq 0$, and $\Phi_z(z, y) \neq 0$

Let $z_0 > 0$, $y_0 > 0$ (inside the region of convergence) satisfy the system of equations:

$$\boxed{y_0 = \Phi(z_0, y_0), \quad 1 = \Phi_y(z_0, y_0)}.$$

Then there exists analytic function $g(z)$ and $h(z)$ such that locally

$$\boxed{a(z) = g(z) - h(z)\sqrt{1 - \frac{z}{z_0}}},$$

where $g(z_0) = y_0$ and $h(z_0) \neq 0$.

A Single Functional Equation

The case $\Phi_{yy}(z,y) = 0$.

$$y = \Phi(z,0) + \Phi_y(z,0)y$$

$$\boxed{y = a(z) = \frac{\Phi(z,0)}{1 - \Phi_y(z,0)}}$$

$$1 = \Phi_y(z_0,0) \quad \Longrightarrow \quad 1 - \Phi_y(z,0) = K(z)(1 - z/z_0)$$

$$\boxed{a(z) = \frac{\Phi(z,0)}{K(z)(1 - z/z_0)}}$$

$\Longrightarrow$ **Polar singularity**

A Single Functional Equation

Two kinds of asymptotic expansions

- **non-affine equation**

$$a(z) = g(z) - h(z)\sqrt{1 - \frac{z}{z_0}} \quad \Longrightarrow \quad [z^n]\, a(z) \sim c\, n^{-3/2} z_0^{-n}$$

- **affine equation**

$$a(z) = \frac{\Phi(z,0)}{K(z)(1 - z/z_0)} \quad \Longrightarrow \quad [z^n]\, a(z) \sim c\, z_0^{-n}$$

Remark. This only applies if there is only one singularity on the circle of convergence. Otherwise we have a **periodic behaviour of the leading coefficients**.

Several Equations

Example.

$$
\begin{aligned}
y_1 &= z(y_2 + y_1^2)\\
y_2 &= z(y_3 + y_2^2)\\
y_3 &= z(1 + y_3^2)
\end{aligned}
$$

$$
\begin{aligned}
y_1 = a_1(z) &= \frac{1-(1-2z)^{1/8}\sqrt{2z\sqrt{2z\sqrt{1+2z}+\sqrt{1-2z}}+(1-2z)^{3/4}}}{2z}\\
y_2 = a_2(z) &= \frac{1-(1-2z)^{1/4}\sqrt{2z\sqrt{1+2z}+\sqrt{1-2z}}}{2z}\\
y_3 = a_3(z) &= \frac{1-\sqrt{1-4z^2}}{2z}
\end{aligned}
$$

$a_1(x)$ has dominant singularity $\boxed{(1-2z)^{1/8}}$.

Several Equations

Example.

$$y_1 = z(y_2^3 + y_1)$$
$$y_2 = z(1 + 2y_2y_3)$$
$$y_3 = z(1 + y_3^2)$$

$$y_1 = a_1(z) = \frac{z}{1-z}\left(\frac{z}{\sqrt{1-4z^2}}\right)^3$$
$$y_2 = a_2(z) = \frac{z}{\sqrt{1-4z^2}}$$
$$y_3 = a_3(z) = \frac{1-\sqrt{1-4z^2}}{2z}$$

$a_1(x)$ has dominant singularity $\boxed{(1-2z)^{-3/2}}$.

Several Equations

Theorem 1

Suppose that $\mathbf{y} = \mathbf{P}(z, \mathbf{y})$ is a well defined $\mathbb{R}_+$-algebraic system of equations with solution $(a_1(z), \ldots, a_d(z))$.

Then all functions $a_j(z)$ have a **finite radius of convergence** ρ_j and for every j we either have

$$\boxed{a_j(z) = c_{0,j} + c_{1,j}(1 - z/\rho_j)^{2^{-k_j}} + c_{2,j}(1 - z/\rho_j)^{2 \cdot 2^{-k_j}} + \ldots}$$

for an integer $k_j \ge 1$ (and $c_{0,j} \ne 0$) or

$$\boxed{a_j(z) = \frac{c_{-m_j,j}}{(1 - z/\rho_j)^{m_j 2^{-k_j}}} + \frac{c_{-m_j+1,j}}{(1 - z/\rho_j)^{(m_j-1)2^{-k_j}}} + \ldots}.$$

for integer $k_j \ge 0$ and $m_j \ge 1$ (and $c_{-m_j,j} \ne 0$).

Several Equations

Theorem 1 (cont.)

For every j there exists $m_j \ge 1$ and for every residue class $k \bmod m_j$ we either have

$$\boxed{a_{j;n} = [z^n]\, a_j(z) = 0} \qquad (n \equiv k \bmod m_j, n \ge n_0)$$

or there exist $\rho_{j,k} > 0$, $c_{j,k} > 0$ and $\boxed{\alpha_{j,k} \in E}$ with

$$\boxed{a_{j;n} = [z^n]\, a_j(z) \sim c_{j,k}\, n^{\alpha_{j,k}}\, \rho_{j,k}^{-n}} \qquad (n \equiv k \bmod m_j,\ n \to \infty),$$

where

$$E = \{-2^{-k} - 1 : k \ge 1\} \cup \{m2^{-k} - 1 : m \ge 1, k \ge 0\}.$$

Systems of functional equations

Dependency graph: $G_{\mathbf{P}} = (V, E)$

V ... vertex set $= \{y_1, y_2, \ldots, y_r\}$
E ... (directed) edge set:

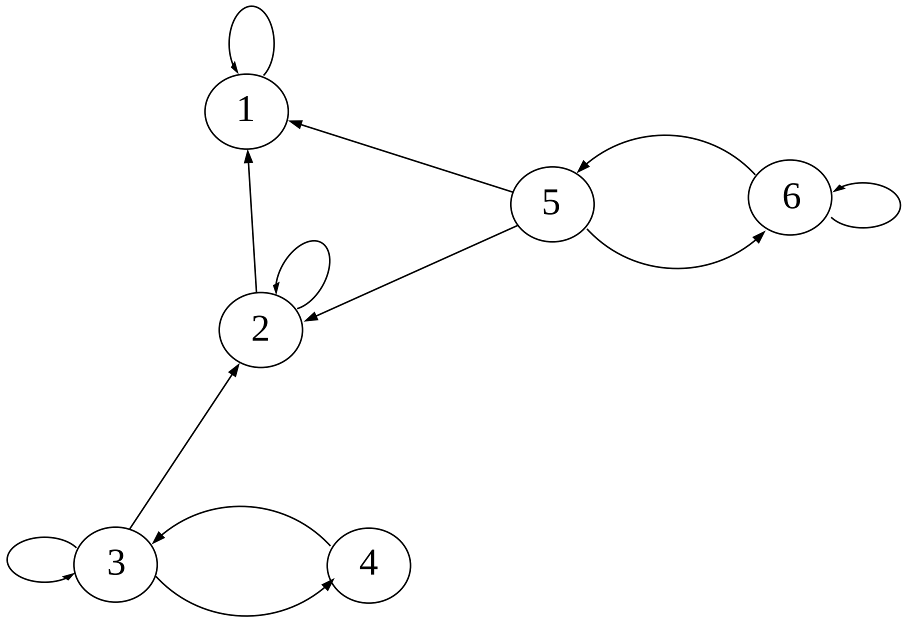
$$\begin{aligned}(y_i, y_j) \in E :&\Longleftrightarrow a_j(z) \text{ depends on } a_i(z)\\ &\Longleftrightarrow P_j \text{ depends on } y_i\\ &\Longleftrightarrow \frac{\partial P_j}{\partial y_i} \neq 0.\end{aligned}$$

Stongly connected dependency graphs.

$$\begin{aligned}G_{\mathbf{P}} \text{ is strongly connected} &\Longleftrightarrow \mathbf{P_y} := \left(\frac{\partial P_j}{\partial y_i}\right) \text{ irreducible}\\ &\Longleftrightarrow \text{no subsystem can be solved}\\ & \text{ before the whole system}\end{aligned}$$

Systems of functional equations

Dependency Graph.



$$y_1 = P_1(z, y_1, y_2, y_5)$$
$$y_2 = P_2(z, y_2, y_3, y_5)$$
$$y_3 = P_3(z, y_3, y_4)$$
$$y_4 = P_4(z, y_3)$$
$$y_5 = P_5(z, y_6)$$
$$y_6 = P_6(z, y_5, y_6)$$

Systems of functional equations

Theorem [D., Lalley, Woods]

Suppose that $\mathbf{y} = \mathbf{\Phi}(z, \mathbf{y})$ is a **positive** and **non-affine** system. Suppose further, that the **dependency graph** of the system $\mathbf{y} = \mathbf{\Phi}(z, \mathbf{y})$ is **strongly connected**.

Let $z_0 > 0$, $\mathbf{y}_0 = (y_{0,0}, \ldots, y_{r,0}) > 0$ (inside the region of convergence) satisfy the system of equations: $(\mathbf{\Phi} = (\Phi_1, \ldots, \Phi_r))$

$$\boxed{\mathbf{y}_0 = \mathbf{\Phi}(z_0, \mathbf{y}_0), \quad 0 = \det(\mathbf{I} - \mathbf{\Phi}_{\mathbf{y}}(z_0, \mathbf{y}_0))}$$

such that all eigenvalues of $\mathbf{\Phi}_{\mathbf{y}}(z_0, \mathbf{y}_0)$ have modulus ≤ 1.

Then there exists analytic function $g_j(z), h_j(z) \ne 0$ such that locally

$$\boxed{y_j(z) = g_j(z) - h_j(z)\sqrt{1 - \frac{z}{z_0}}}.$$

Systems of functional equations

The affine case $\Phi_{\mathbf{yy}}(z, \mathbf{y}) = 0$.

$$\mathbf{y} = \mathbf{\Phi}(z, 0) + \mathbf{\Phi}_{\mathbf{y}}(z, 0)\mathbf{y}$$

$$\boxed{\mathbf{y} = \mathbf{a}(z) = \frac{H(z)}{\det(\mathbf{I} - \mathbf{\Phi}_{\mathbf{y}}(z, 0))}}$$

$G_{\mathbf{\Phi}}$ strongly connected $\Longrightarrow$ $\mathbf{\Phi}_{\mathbf{y}}$ irreducible
$\Longrightarrow$ there is a simple dominant root $z_0 > 0$ of

$$z \mapsto \det(\mathbf{I} - \mathbf{\Phi}_{\mathbf{y}}(z, 0))$$

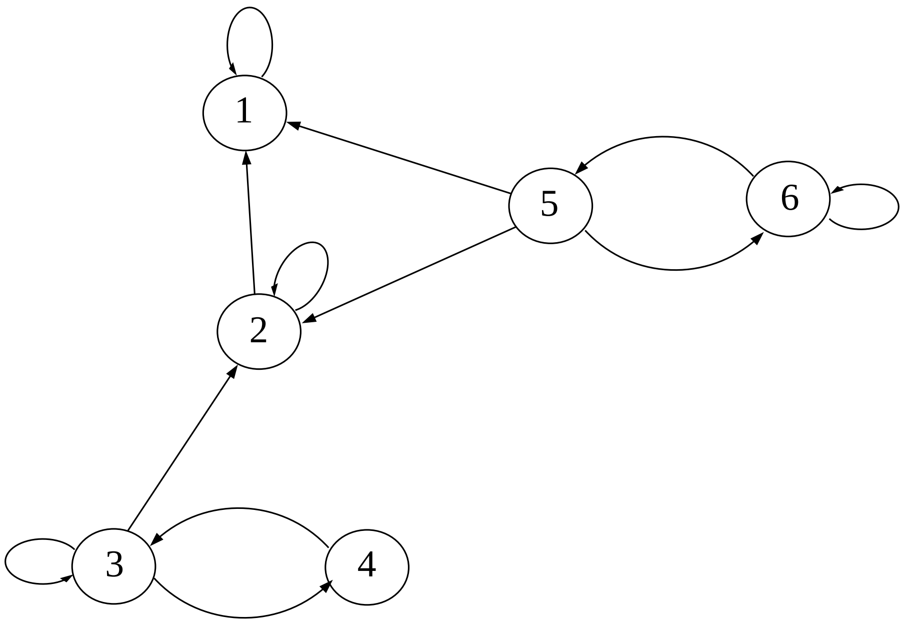
$\Longrightarrow$ **Polar singularity** at z_0

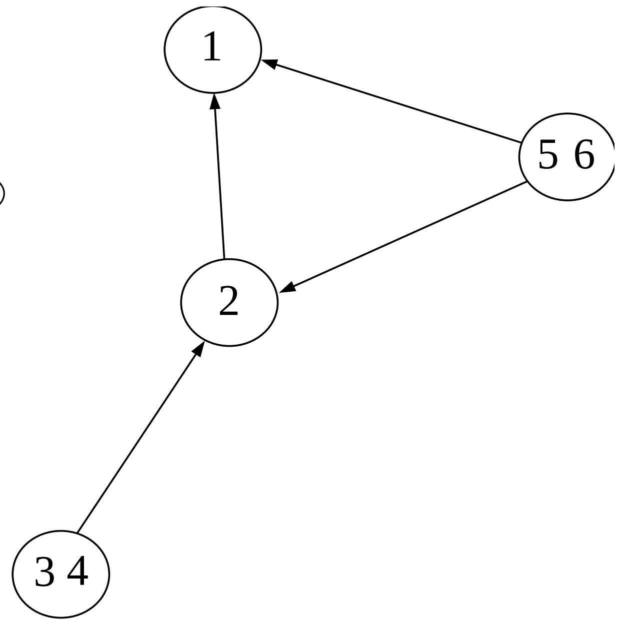
General Dependency Graphs

Dependency Graph and Reduced Dependency Graph

$$
\begin{aligned}
y_1 &= P_1(z, y_1, y_2, y_5) \\
y_2 &= P_2(z, y_2, y_3, y_5) \\
y_3 &= P_3(z, y_3, y_4) \\
y_4 &= P_4(z, y_3) \\
y_5 &= P_5(z, y_6) \\
y_6 &= P_6(z, y_5, y_6)
\end{aligned}
$$

$$
\begin{aligned}
y_1 &= P_1(z, y_1, y_2, (y_5, y_6)) \\
y_2 &= P_2(z, y_2, (y_3, y_4), (y_5, y_6)) \\
(y_3, y_4) &= (P_3, P_4)(z, y_3, y_4) \\
(y_5, y_6) &= (P_5, P_6)(z, y_5, y_6)
\end{aligned}
$$





General Depencency Graphs

Two equations

$$y_1 = P_1(z, y_1, y_2)$$
$$y_2 = P_2(z, y_2)$$

$$\Longrightarrow \quad y_2 = a_2(z) = g_2(z) - h_2(z)\sqrt{1 - z/\rho_2},$$
$$y_1 = y_1(z, y_2) = g_1(z, y_2) - h_1(z, y_2)\sqrt{1 - z/\rho(y_2)}$$

$$\begin{aligned}\Longrightarrow \quad a_1(z) &= y_1(z, a_2(z)) \\ &= g_1(z, a_2(z)) - h_1(z, a_2(z))\sqrt{1 - z/\rho(a_2(z))} \\ &= g_1(z, a_2(z)) - h_1(z, a_2(z))\rho(a_2(z))^{-1/2}\sqrt{\rho(a_2(z)) - z}\end{aligned}$$

3 cases: (1) $\rho(a_2(\rho_2)) > \rho_2$ (2) $\rho(a_2(\rho_2)) = \rho_2$ (3) $\rho(a_2(\rho_2)) < \rho_2$

General Depencency Graphs

Case (1). $\boxed{\rho(a_2(\rho_2)) > \rho_2}$

$$
\begin{aligned}
g_1(z, a_2(z)) &= g_1\left(z, g_2(z) - h_2(z)\sqrt{1 - z/\rho_2}\right) \\
&= g_1(\rho_2, g_2(\rho_2)) - g_{1,y}(\rho_2, g_2(\rho_2))h_2(\rho_2)\sqrt{1 - z/\rho_2}) + \cdots \\
h_1(z, a_2(z)) &= h_1(\rho_2, g_2(\rho_2)) - h_{1,y}(\rho_2, g_2(\rho_2))h_2(\rho_2)\sqrt{1 - z/\rho_2} + \cdots \\
\rho(a_2(z)) - z &= \rho(a_2(\rho_2)) - \rho_2 - \rho'(a_2(\rho_2))h_2(\rho_2)\sqrt{1 - z/\rho_2} + \cdots \\
\sqrt{\rho(a_2(z)) - z} &= \sqrt{\rho(a_2(\rho_2)) - \rho_2 - \rho'(a_2(\rho_2))h_2(\rho_2)\sqrt{1 - z/\rho_2}) + \cdots} \\
&= \sqrt{\rho(a_2(\rho_2)) - \rho_2} - c_1\sqrt{1 - z/\rho_2} + \cdots
\end{aligned}
$$

$$
\begin{aligned}
\Longrightarrow \quad a_1(z) &= g_1(z, a_2(z)) - h_1(z, a_2(z))\rho(a_2(z))^{-1/2}\sqrt{\rho(a_2(z)) - z} \\
&= c_0 - c_1\sqrt{1 - z/\rho_2} + \cdots
\end{aligned}
$$

General Depencency Graphs

Case (2). $\boxed{\rho(a_2(\rho_2)) = \rho_2}$

$$\begin{aligned}
\rho(a_2(z)) - z &= \rho(a_2(\rho_2)) - \rho_2 - \rho'(a_2(\rho_2))h_2(\rho_2)\sqrt{1 - z/\rho_2}) + \cdots \\
&= c_1'\sqrt{1 - z/\rho_2} + \cdots \\
\sqrt{\rho(a_2(z)) - z} &= \sqrt{c_1'\sqrt{1 - z/\rho_2}) + \cdots} \\
&= \sqrt{c_1'}(1 - z/\rho_2)^{1/4} + c_2'(1 - z/\rho_2)^{3/4} + \cdots
\end{aligned}$$

$$\begin{aligned}
\Longrightarrow \quad a_1(z) &= g_1(z, a_2(z)) - h_1(z, a_2(z))\rho(a_2(z))^{-1/2}\sqrt{\rho(a_2(z)) - z} \\
&= c_0 + c_1(1 - z/\rho_2)^{1/4} + c_2\sqrt{1 - z/\rho_2} + \cdots
\end{aligned}$$

General Depencency Graphs

Case (3). $\boxed{\rho(a_2(\rho_2)) < \rho_2}$

There exists $\rho_1 < \rho_2$ with $\rho(a_2(\rho_1)) = \rho_1$:

$$\begin{aligned} \rho(a_2(z)) - z &= \rho(a_2(\rho_1)) - \rho_1 + \rho'(a_2(\rho_1))a_2'(\rho_1)(z - \rho_1) \\ &= c_1''(\rho_1 - z) + \cdots \\ \sqrt{\rho(a_2(z)) - z} &= \sqrt{c_1''}\sqrt{\rho_1 - z} + \cdots \\ &= \sqrt{c_1''\rho_1}\sqrt{1 - z/\rho_1} + \cdots \end{aligned}$$

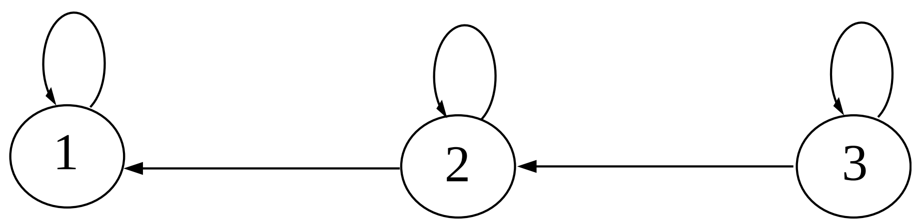
$$\Longrightarrow \quad \begin{aligned} a_1(z) &= g_1(z, a_2(z)) - h_1(z, a_2(z))\rho(a_2(z))^{-1/2}\sqrt{\rho(a_2(z)) - z} \\ &= c_0 - c_1\sqrt{1 - z/\rho_1} + \cdots \end{aligned}$$

with $\rho_1 < \rho_2$.

Systems of functional equations

Counter-Example

$$y_1 = z(e^{y_2} + y_1)$$
$$y_2 = z(1 + 2y_2 y_3)$$
$$y_3 = z(1 + y_3^2)$$



$$y_1 = a_1(z) = \frac{z}{1-z}\exp\left(\frac{z}{\sqrt{1-4z^2}}\right)$$
$$y_2 = a_2(z) = \frac{z}{\sqrt{1-4z^2}}$$
$$y_3 = a_3(z) = \frac{1-\sqrt{1-4z^2}}{2z}$$

General Depencency Graphs

Theorem 2

Suppose that $\mathbf{y} = \mathbf{\Phi}(z, \mathbf{y})$ is a **positive** system of entire functions such that there is a unique solution $(a_1(z), \ldots, a_d(z))$ that is analytic at $z = 0$.

Then all functions $a_j(z)$ have **non-negative coefficients** and a **finite radius of convergence** ρ_j.

(A) If $\boxed{\frac{\partial^2 \Phi_j}{\partial y_j^2} \neq 0}$ (for all j) then for every j there exists an integer $k_j \geq 1$ such that locally

$$\boxed{a_j(z) = c_{0,j} + c_{1,j}(1 - z/\rho_j)^{1/2^{k_j}} + c_{2,j}(1 - z/\rho_j)^{2/2^{k_j}} + \ldots}.$$

General Depencency Graphs

Theorem 2 (cont.)

(B) If we just have the condition that for all pairs (i,j) with $\frac{\partial \Phi_j}{\partial y_i} \neq 0$ there exists k with $\boxed{\frac{\partial^2 \Phi_j}{\partial y_i y_k} \neq 0}$ then for every j we either have

$$\boxed{a_j(z) = c_{0,j} + c_{1,j}(1 - z/\rho_j)^{2^{-k_j}} + c_{2,j}(1 - z/\rho_j)^{2 \cdot 2^{-k_j}} + \dots}$$

for an integer $k_j \geq 1$ or

$$\boxed{a_j(z) = \frac{c_{-1,j}}{(1 - z/\rho_j)^{2^{-k_j}}} + c_{0,j} + c_{1,j}(1 - z/\rho_j)^{2^{-k_j}} + \dots}.$$

for an integer $k_j \geq 0$.

Infinite Systems

Infinite linear systems. $\mathbf{y} = A(z)\mathbf{y} + \mathbf{b}(z) \Longrightarrow \boxed{\mathbf{y}(\mathbf{z}) = (\mathbf{I} - A(z))^{-1}\mathbf{b}(z)}$

Example.

$$y_1 = 1 + zy_2$$
$$y_j = z(y_{j-1} + y_{j+1})$$

$$\Longrightarrow \quad \boxed{y_j = a_j(z) = \frac{1}{z}\left(\frac{1 - \sqrt{1 - 4z^2}}{2z}\right)^j}$$

$$A = \begin{pmatrix} 0 & z & 0 & 0 & \cdots \\ z & 0 & z & 0 & \cdots \\ 0 & z & 0 & z & \\ 0 & 0 & z & \ddots & \ddots \\ \vdots & \vdots & \vdots & & \ddots \end{pmatrix}$$

Infinite Systems

Example.

$$y_0 = 1 + z^2 y_0 + z y_1$$
$$y_1 = z(1+z) y_0 + z y_2$$
$$y_j = z(y_{j-1} + y_{j+1})$$

$$\Longrightarrow \quad \boxed{y_j = a_j(z) = \frac{2}{\sqrt{1-2z}\left(\sqrt{1+2z} - \sqrt{1-2z}\right)} \left(\frac{1 - \sqrt{1-4z^2}}{2z} \right)^{j+1}} \quad (j \geq 1$$

$$A = \begin{pmatrix} z^2 & z & 0 & 0 & \cdots \\ z(1+z) & 0 & z & 0 & \cdots \\ 0 & z & 0 & z & \\ 0 & 0 & z & \ddots & \ddots \\ \vdots & \vdots & \vdots & & \ddots \end{pmatrix}$$

Infinite Systems

Compact operator $A(z)$. $\quad \mathbf{y} = A(z)\mathbf{y} + \mathbf{b}(z)$

$A(z)$ ... irreducible (and compact in a proper ℓ^p-space)
$r(A(z))$ ... spectral radius of $A(z)$.

$r(A(z_0)) = 1 \Longrightarrow$ resolvent $(x\mathbf{I} - \mathbf{A}(\mathbf{z_0}))^{-\mathbf{1}}$ has a simple pole
$\Longrightarrow \mathbf{y} = \mathbf{a}(\mathbf{z}) = (\mathbf{I} - A(z))^{-1}\mathbf{b}(z)$ has a **simple pole** at $z = z_0$.

This is the same situation as in the finite dimensional case

Infinite Systems

Theorem [Lalley, Morgenbesser]

Suppose that $\mathbf{y} = (y_j)_{j\ge 1} = \mathbf{\Phi}(z,\mathbf{y})$ is a **positive**, **non-linear**, **infinite** and **irreducibe** system such that $\mathbf{\Phi}_{\mathbf{y}}(z,\mathbf{y})$ is **compact**.

Let $z_0 > 0$, $\mathbf{y}_0 = (y_{0,0},\ldots,y_{r,0}) > 0$ (inside the region of convergence) satisfy the system of equations: $(\mathbf{\Phi} = (\Phi_1,\ldots,\Phi_r))$

$$\boxed{\mathbf{y}_0 = \mathbf{\Phi}(z_0,\mathbf{y}_0), \quad r(\mathbf{\Phi}_{\mathbf{y}}(z_0,\mathbf{y}_0)) = 1}.$$

Then there exists analytic function $g_j(z), h_j(z) \ne 0$ such that locally

$$\boxed{a_j(z) = g_j(z) - h_j(z)\sqrt{1-\frac{z}{z_0}}}.$$

with $g_j(z_0) = (\mathbf{y}_0)_j$ and $h_j(z_0) \ne 0$.

Thank You!