Partitions with Distinct Multiplicities of Parts: On An "Unsolved Problem" Posed By Herbert Wilf

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WILF'S SIXTH "UNSOLVED PROBLEM"

In the later years of his life, Herbert S. Wilf (1931–2012) posted a set of eight Unsolved Problems on his webpage. At the time of Herb's death, only one of these problems was solved [the third problem, about an asymptotic rational approximation to π, by Mark Ward (2010)]. This talk concerns Wilf's sixth "Unsolved Problem":

Distinct multiplicities Wilf (2010)

Let $T(n)$ be the set of partitions of n for which the (nonzero) multiplicities of its parts are all different, and write $f(n) = |T(n)|$. See Sloane's sequence `A098859` for a table of values. Find any interesting theorems about $f(n)$. The mapping that sends a partition of n to another partition of n in which the roles of parts and multiplicities are interchanged is a well defined involution on $T(n)$, which is how I arrived at the study of this problem.

DEFINITIONS

Wilf partitions are partitions in which the (nonzero) multiplicities of the parts are all different:

- multiplicities $\mathcal{M}_r := \{\mathbf{m} = (m_1, m_2, \ldots, m_r) :$ m_k are distinct positive integers$\}$
- parts $\mathcal{P}_r := \{\mathbf{p} = (p_1, p_2, \ldots, p_r) :$ p_k are distinct positive integers with $p_1 < \cdots < p_r\}$
- Wilf partitions of n: $T(n) := \bigcup_{r\geq 1}\{(\mathbf{m}, \mathbf{p}) \in \mathcal{M}_r \times \mathcal{P}_r : m_1p_1 + \cdots + m_rp_r = n\}$
- $p(n, r) :=$ number of partitions of n into r parts
- divisor function $d(\cdot)$: $d(n)$ is the number of divisors of n
- $f(n) := |T(n)|$; Wilf: Find anything interesting about $f(n)$.

MAIN RESULT

Theorem

Let $f(n)$ denote the number of Wilf partitions of n. Then

$$\ln f(n) \sim \frac{6^{1/3}}{3} n^{1/3} \ln n \sim (6n)^{1/3} \ln[(6n)^{1/3}] \qquad \text{as } n \to \infty.$$

The theorem will be established by matching upper and lower bounds for $\ln f(n)$.

Remark

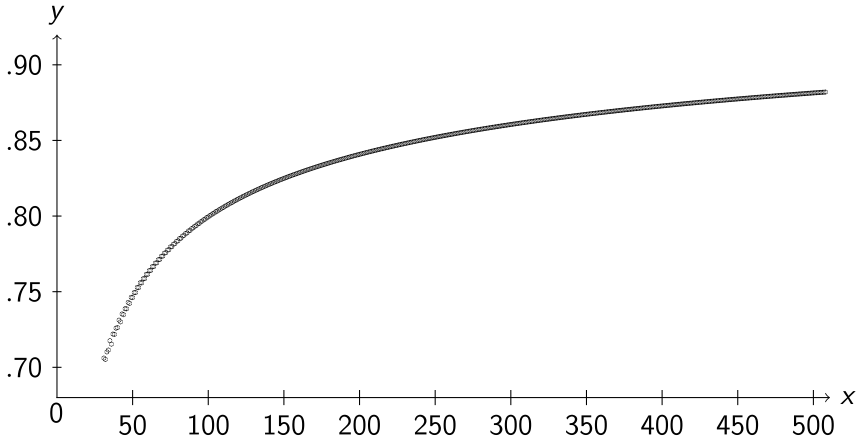
Independently, Daniel Kane and Robert C. Rhoades, Stanford Math, 2012, unpublished) have proven the following improvement:

$$\ln f(n) = \frac{6^{1/3}}{3} n^{1/3} \ln n - \frac{6^{1/3}}{2} n^{1/3} \ln \ln n + O(n^{1/3}).$$

MAIN RESULT

Here is a plot of the values of

$$\frac{\ln f(n)}{(6n)^{1/3}\ln[(6n)^{1/3}]} \quad \text{for } 31 \leq n \leq 508:$$



Lemma Used in Proof of Upper Bound

Lemma

The number r of distinct multiplicities in a Wilf partition of n is at most $(6n)^{1/3}$.

Proof.

For a given positive integer r, the smallest possible n admitting a Wilf partition with r distinct multiplicities is obtained by taking:

- multiplicity $m_1 = r$ for part $p_1 = 1$;
- multiplicity $m_2 = r - 1$ for part $p_2 = 2$;
- multiplicity $m_3 = r - 2$ for part $p_3 = 3$;
- $\vdots$
- multiplicity $m_r = 1$ for part $p_r = r$.

This yields

$$n = \sum_{i=1}^{r}(r+1-i)i = \frac{1}{6}r^3 + \frac{1}{2}r^2 + \frac{1}{3}r.$$

Hence $r \leq (6n)^{1/3}$. □

Proof of Upper Bound

Lemma

An upper bound for $\ln f(n)$ *is*

$$\ln f(n) \leq (1 + o(1))\frac{6^{1/3}}{3} n^{1/3} \ln n.$$

Proof.

- For each Wilf partition of n, put the terms in decreasing order according to the values of the products $m_i p_i$.
- If two terms are equal, break the tie by writing in decreasing order of the multiplicities.
- This gives canonical way to write Wilf partitions. For instance,

  $$27 = 8 + 3 + 3 + 2 + 2 + 2 + 1 + 1 + 1 + 1 + 1 + 1 + 1$$

  is written as

  $$27 = (1\times 8) + (7\times 1) + (3\times 2) + (2\times 3) = m_1 p_1 + m_2 p_2 + m_3 p_3 + m_4 p_4.$$

Proof of Upper Bound (continued)

- It follows that the number $f(n, r)$ of Wilf partitions of n with r distinct multiplicities is no larger than

$$p(n, r) \times [\max\{d(j) : 1 \leq j \leq n\}]^r.$$

- Severin Wigert (1907) (see also Hardy and Wright, 1960, Th. 317, Chap. XVIII.1) showed that

$$\limsup_{n \to \infty} \frac{\ln d(n)}{(\ln n)/(\ln \ln n)} = \ln 2.$$

- Therefore, there exists a constant C such that, provided n is sufficiently large,

$$\begin{aligned} f(n, r) &\leq p(n, r) \times \max_{3 \leq j \leq n} \left\{ \exp\left(rC \frac{\ln j}{\ln \ln j} \right) \right\} \\ &= p(n, r) \times \exp\left(rC \frac{\ln n}{\ln \ln n} \right). \end{aligned} \quad (1)$$

Proof of Upper Bound (conclusion)

- Since by the preceding lemma we have $r \leq (6n)^{1/3}$ if $f(n,r) > 0$, the second factor here does not contribute to the lead-order logarithmic asymptotics for $f(n)$.
- Now we utilize Exercise 7.2.1.4-34 in Knuth (2011, Volume 4A), which concerns $p(n,r)$ and is stated (in our notation) for $r \leq n^{1/3}$; but Knuth's argument is easily checked to hold also for $r \leq (cn)^{1/3}$ for any constant c. Choosing $c = 6$ we find, for all $r \leq (6n)^{1/3}$, that, as $n \to \infty$, we have

$$p(n,r) = O\left(\frac{n^{r-1}}{r!(r-1)!}\right) \leq \exp\left[(1+o(1))(1/3)(6n)^{1/3} \ln n\right]$$

- Then (1) on preceding page yields same estimate for $f(n,r)$.
- The upper-bound proof is completed by a summation over $r \leq (6n)^{1/3}$. □

Proof of Lower Bound

Lemma

A lower bound for $\ln f(n)$ *is*

$$\ln f(n) \geq (1+o(1))\frac{6^{1/3}}{3} n^{1/3} \ln n.$$

Proof.

- Let $a < 6^{1/3}$, and let K be a fixed large integer. Let $b := \lfloor an^{1/3}/K \rfloor$ and divide the interval $[1, Kb] \subseteq [1, an^{1/3}]$ into K equal parts $I_1, \ldots, I_K$.
- Consider only permutations $(p_1, \ldots, p_{Kb})$ of $[1, Kb]$ that map I_j into I_{K+1-j} for every $j \in \{1, \ldots, K\}$.
- For such permutations, if $a = [6(1-2\epsilon)]^{1/3}$, and if K is large enough (depending on ϵ but not on n), then $\sum_i ip_i < (1-\epsilon)n$, and we obtain (if n is large enough) a Wilf partition by taking i parts of size p_i for each $2 \leq i \leq Kb$ and a single part of size $n - \sum_{i=2}^{Kb} ip_i$.

Proof of Lower Bound (conclusion)

- Thus the number of Wilf partitions of n is at least

$$b!^K = \exp\left[Kb(\ln b + O(1))\right] = \exp\left[an^{1/3}(\ln n^{1/3} + O(1))\right]$$
$$= \exp\left[\left(\frac{a}{3} + o(1)\right) n^{1/3} \ln n\right].$$

This completes the lower-bound proof, since we may take a arbitrarily close to $6^{1/3}$. □

OPEN PROBLEMS

- Find further terms in asymptotic expansion of $\ln f(n)$.
- Herculean task: Find lead-order asymptotics for $f(n)$.
- David S. Newman has mentioned that it would be nice to have a proof that $f(n)$ is nondecreasing.
- Study the number of fixed points for an involution: ...

Open Problem: An Involution

- Wilf mentions a mapping σ_n on $T(n)$ in which the roles of parts and multiplicities are interchanged.
- Hence, $\sigma_n((\mathbf{m}, \mathbf{p}))$ equals

$$(\mathbf{p}_\pi, \mathbf{m}_\pi) = ((p_{\pi(1)}, \ldots, p_{\pi(r)}), (m_{\pi(1)}, \ldots, m_{\pi(r)})),$$

  where π is the permutation making $m_{\pi(1)} < \cdots < m_{\pi(r)}$.
- For instance, two partitions of 83 are

$$\begin{aligned}83 &= 1+1+1+1+1+1+1+4+4+4+4\\ &\quad +5+5+5+5+5+5+5+5+5+5+5+5\\ &= ((7,4,12),(1,4,5))\end{aligned}$$

  and

$$\begin{aligned}83 &= 4+4+4+4+7+12+12+12+12+12\\ &= ((4,1,5),(4,7,12)).\end{aligned}$$

Open Problem: An Involution (continued)

- Then we have

$$\sigma_{83} : ((7,4,12),(1,4,5)) \mapsto ((4,1,5),(4,7,12)),$$
$$\sigma_{83} : ((4,1,5),(4,7,12)) \mapsto ((7,4,12),(1,4,5)).$$

  The mapping σ_n is an involution.

- Note that σ_n has order 2 on most elements of $T(n)$, but σ_n has order 1 on some elements. In particular, σ_n fixes every $(\mathbf{m}, \mathbf{p})$ for which $m_i = p_i$ for all i, for example,

$$\sigma_{65} : ((2,5,6),(2,5,6)) \mapsto ((2,5,6),(2,5,6));$$

  but there are also other examples, such as

$$\sigma_{10} : ((1,2,3),(3,2,1)) \mapsto ((1,2,3),(3,2,1)).$$

- Open problem: Find (asymptotics for) the number of fixed points of σ_n.