Part Sizes of Smooth Supercritical Compositional Structures

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Based on joint work with Ed Bender

A simple example

Ordinary compositions:

12	32	2	1	3	…	320	45	32

⇧

a positive integer

Supports are the array of boxes, and the *parts* are the positive integers.

$$\textit{support generating function } S(x) = \sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$$

$$\textit{part generating function } P(x) = \sum_{k=1}^{\infty} x^k = \frac{x}{1-x}$$

$$\textit{composition generating function } S(P(x)) = \frac{1-x}{1-2x}$$

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- local restrictions: parts within a fixed window satisfy certain constraints. For example, adjacent parts are distinct (Carlitz restrictions); Any three consecutive parts don't form a Pythagorean triple.
- matrix compositions: supports are $r \times m$ rectangles where r is a fixed positive integer. (Louchard, 08)

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Let $g_{n,k} = [x^n]S^{(k)}(P(x))$. So $g_{n,0} = [x^n]S(P(x))$. We call the family *smooth supercritical* if it is supercritical and satisfies the following *smoothness* conditions:

(a) There is a constant $\delta > 0$ such that $g_{n,0}/g_{n+t,0} \to r^t$ uniformly for $|t| \leq n^{\delta}$.

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We note that if both $P(x)$ and $S(x)$ are of "algebraic-logarithmic" type, then the family satisfies the above smoothness conditions.

Our main results

Notation: $\mathbb{N} = \{1, 2, \ldots, \}$, $\mathcal{P} = \{i : p_i > 0\}$, $\gamma \doteq 0.577216$ denotes Euler's constant, $\omega(n)$ denotes any function going to ∞ as $n \to \infty$.

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Let $r \in (0, \rho(P))$ be defined by $P(r) = \rho(S)$ and $\alpha = \rho(P)/r$. Let $\log$ denote logarithm to the base α, and let

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Theorem (Main Results)

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Let $\sigma(n)$ be given by $\alpha^{\sigma(n)} e^{-f(\sigma(n))} = n/P'(r)$.

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(c) *There is an $\epsilon > 0$ such that $s_k \leq \exp(O(k^{1-\epsilon}))\rho(S)^{-k}$ for all k, and there is an infinite set $K = \{k_1 < k_2 < \cdots\} \subseteq \mathbb{N}$ such that*

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Examples

Hypercubes. For d-dimensional hypercubes, $s_k = 1$ if k is a dth power and $s_k = 0$ otherwise. In this case, we let
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Ferrer's diagrams. When the supports are Ferrer's diagrams, s_k is the number of partitions of k and we have $s_k \sim \exp(c_1\sqrt{n} - \ln n + c_0)$. So we can take $K = \mathbb{N}$.

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If the alphabet Σ has N letters, then $P(x) = \frac{x}{1-x}$, $S(x) = \frac{Nx}{1-(N-1)x}$, and $S(P(x)) = \frac{Nx}{1-Nx}$.

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$$\mathrm{E}(M_n) = \sigma(n) + \gamma \log e - \frac{1}{2} + P_0(\sigma(n)) + o(1).$$

Runs of a particular letter. Let $M_n(j)$ be the maximum run of a particular letter j. We note $M_n = \max\{M_n(j) : j \in \Sigma\}$.

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Theorem

Let ζ_j be the number of parts of size j in a random structure of size n. Then there is a function $\omega(n) \to \infty$ such that the random variables $\{\zeta_j : \sigma(n) - \omega(n) \leq j \leq n\}$ are asymptotically independent Poisson random variables with means $\mu_j = \alpha^{\sigma(n)-j}$.

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THANK YOU

Some references

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