Associative and commutative tree representations for Boolean functions

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AofA 2013, Menorca, May 29, 2013

* Supported by the Austrian Science Foundation FWF, SFB F50

And/Or trees and the limiting probability

Take

- a set of Boolean connectors $\{\wedge, \vee\}$
- a set of literals $\{x_1, \bar{x}_1, \ldots, x_n, \bar{x}_n\}$
- a family $\mathcal{T}$ of unlabelled trees, size $m = \#$ leaves

And/Or tree: element $t \in \mathcal{T}$

- internal vertices labelled with connectors
- leaves labelled with literals

$\Rightarrow$ represents a Boolean function $f : \{0, 1\}^n \rightarrow \{0, 1\}$

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limiting probability of f

$$\mathbb{P}_n(f) = \lim_{m\to\infty} \mathbb{P}_{m,n}(f) = \lim_{m\to\infty} \frac{T_{m,n}(f)}{T_{m,n}} \qquad \text{(if it exists)}$$

Previous work

- Catalan model (And/Or)
  - Lefmann, Savický ’97
  - Chauvin, Gardy, Flajolet, G. ’04
  - Gardy ’06
  - Kozik ’08
- Catalan model (implication)
  - Fournier, Gardy, Genitrini, G. ’08, ’12
  - Genitrini, G., Kraus, Mailler ’12

Models considered

- classical (Catalan) model: binary, planar
- associative model: outdegree $\neq 1$, planar
  ”stratified”: neighbours cannot have the same label
- commutative model: binary, non-plane
- general (Pólya) model: outdegree $\neq 1$, nonplane, stratified
  (=associative & commutative)

All limiting probabilities exist

Lemma

In all models, the limiting probability $\mathbb{P}_n(f)$ exists for all f.

Proof idea: Choose a model,

- $T(z) = \sum_{m \geq 0} T_{m,n} z^m$ generating function of trees
  singularity ρ_n
- $T_f(z) = \sum_{m \geq 0} T_{m,n}(f) z^n$ GF of trees computing f

system of equations for $T_f(z) \Rightarrow$ Drmota-Lalley-Woods theorem: same singularity $\Rightarrow$ transfer lemma.

Example: Associative plane trees

$$A(z) = \hat{A}(z) + \check{A}(z) - 2nz$$

$$\hat{A}(z) = 2nz + \sum_{k\geq 2} \check{A}(z)^k \overset{\check{A}=\hat{A}}{=} 2nz + \frac{\hat{A}^2(z)}{1-\hat{A}(z)}$$

$$\Rightarrow A(z) = \frac{1}{2}\left(1 - 2nz - \sqrt{1 - 12nz + 4n^2z^2}\right)$$

singularity: $\rho_n = \frac{3-2\sqrt{2}}{2n}$.

$$\hat{A}_f(z) = z 1\!\!1_{\{f \text{ lit.}\}} + \sum_{i=2}^{\infty} \sum_{\substack{g_1,\ldots,g_i,\\ g_1\wedge\cdots\wedge g_i = f}} \check{A}_{g_1}(z)\cdots\check{A}_{g_i}(z)$$

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Limiting probabilities

Theorem (Kozik, 2008)

Let $\mathcal{T}$ be the family of binary planar And/Or trees. Then

$$\mathbb{P}_n(f) \sim \frac{\lambda_f}{n^{L(f)+1}},$$

as $n \to \infty$.

Theorem (Fournier, Gardy, Genitrini, G., 2008)

Let $\mathcal{T}$ be the family of binary planar implication trees. Then

$$\mathbb{P}_n(f) \sim \frac{\tilde{\lambda}_f}{n^{L(f)+1}},$$

as $n \to \infty$.

The limiting probabilities of constant functions

Theorem

The limiting probability of constant functions in the different models is

binary plane: $\mathbb{P}_n(\mathit{True}) = \frac{3}{4n} + \mathcal{O}\left(\frac{1}{n^2}\right)$	*associative:* $\mathbb{P}_n(\mathit{True}) = \frac{51-36\sqrt{2}}{n} + \mathcal{O}\left(\frac{1}{n^2}\right)$
binary commutative: $\mathbb{P}_n(\mathit{True}) = \frac{641}{1024n} + \mathcal{O}\left(\frac{1}{n^2}\right)$	*general:* $\mathbb{P}_n(\mathit{True}) = \frac{(2\ln 2-1)^2}{4n} + \mathcal{O}\left(\frac{1}{n^2}\right)$

Limiting distribution of general functions

Theorem

For all models M of And/Or trees studied, and for all Boolean functions f,

$$\mathbb{P}_n(f) \sim \frac{\lambda_M(f)}{n^{L(f)+1}}, \qquad \textit{as } n \to \infty$$

where $\lambda_M(f)$ is related to the $\#$ of possible expansions of a minimal tree of f.

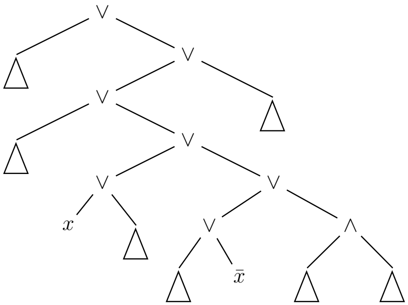
Sketch of proof:

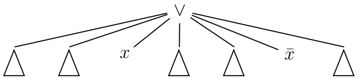
Ingredients:

- Simple tautologies
- Representation of trees by pattern languages
- Expansions by tautologies and literals are enough
- Asymptotically almost every tree computing f is a minimal tree expanded once. (Kozik ’08 for the Catalan model)

Sketch of proof: simple tautologies

simple tautology (realized by x): $x \vee \bar{x} \vee f$





Sketch of proof: pattern languages

pattern language L:

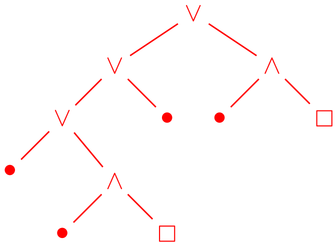
- $*$ planar tree family,
- $*$ internal nodes labelled with connectors,
- $*$ leaves labelled by $\{\bullet, \Box\}$.

together with $\mathcal{T} \Rightarrow L[\mathcal{T}]$:

- $*$ $\Box \leftarrow$ element from $\mathcal{T}$,
- $*$ $\bullet \leftarrow$ label $\in \{x_1, \bar{x}_1, \ldots, x_n, \bar{x}_n\}$.

Example $L[\mathcal{T}]$

$$N = \bullet | N \vee N | N \wedge \square$$



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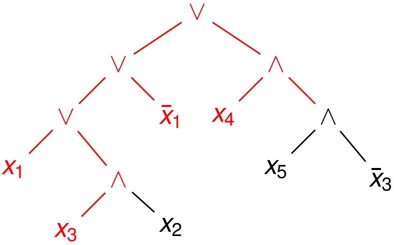

Example $L[\mathcal{T}]$

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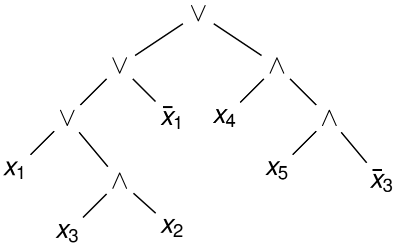

Pattern languages for associative trees

$$R = \{\hat{N}, \check{N}\}$$

$$\hat{N} = \bullet | N \wedge \square | N \wedge \square \wedge \square | \cdots \qquad \check{N} = \bullet | N \vee N | N \vee N \vee N | \cdots$$

- x essential variable of f: f depends on x
- $\#$ repetitions $= \#$ pattern leaves $- \#$ distinct variables
- $\#$ of restrictions $= \#$ repetitions$+\#$ of essential variables

property of N: set all pattern leaves to false $\Rightarrow$ whole tree computes false.

$$\Rightarrow \exists \text{ repetition } x/\bar{x} \text{ in tree computing } \mathit{True}.$$

Simple tautologies

Proposition

If L is subcritical w.r.t. $\mathcal{T}$ then

$$\lim_{m\to\infty} \frac{L[\mathcal{T}]_m^{(k)}}{T_m} = \mathcal{O}\left(\frac{1}{n^k}\right).$$

Lemma

All tautologies with 1 L-restriction are simple.

Proposition

Asymptotically almost all tautologies are simple (and realized by exactly 1 variable), when $m \to \infty$.

Limiting probability of simple tautologies

$ST_x(z)$: gf of simple tautologies realized by x.

$ST(z) = nST_x(z)$.

$$\mathcal{ST}_x = \{\vee\} \times \{x\} \times \{\bar{x}\} \times \texttt{set}(\mathcal{T} \setminus \{x, \bar{x}\})$$

$$ST_x(z) = z^2 \times \sum_{\ell \geq 2} \ell(\ell - 1)(T(z) - 2z)^{\ell-2}$$

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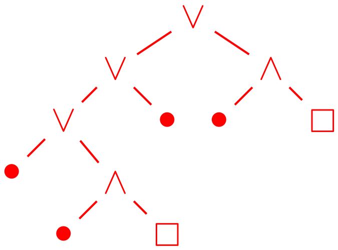
$$\mathbb{P}(\mathit{True}) = \lim_{m \to \infty} \frac{[z^m]ST(z)}{[z^m]T(z)} \sim \lim_{z \to \rho} \frac{ST'(z)}{T'(z)}.$$

Commutative trees

- $\nexists$ unambiguous pattern with $T \mapsto L[T] \mapsto T$.
- “mobiles”: halfembedding of $\mathcal{T}$:
  at each pattern node choose left-right order
  $\Rightarrow$ injection $\mathcal{T} \to L[\mathcal{T}]$
- minimal embedding: choose one with $\#$ restrictions $=$ min.
- No subcriticality any more!

Mobiles

$$N = \bullet | N \vee N | N \wedge \square$$



Mobiles

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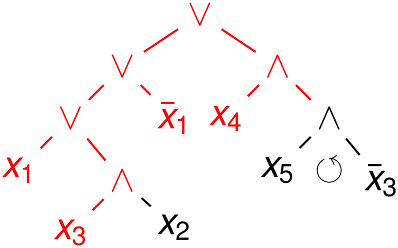

Mobiles

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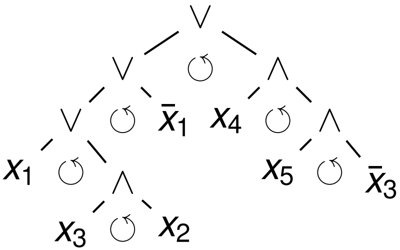

Outlook

- No Shannon effect
  - disproved for Galton-Watson And/Or trees (Lefmann, Savický, 1997)
  - disproved for implication trees (Genitrini, G., 2010)
- Characterize the class of functions which yields the total mass
- define size $=\#$ all vertices in associative models (Work in progress)

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Muchas Gracias!