Combinatorial Markov chains

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$$\mathbb{F}_n := \{x \in \mathbb{F} : \phi(x) = n\}$$
is finite for all $n \in \mathbb{N}$. We assume that $\mathbb{F}_1 = \{e\}$.

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- A combinatorial Markov chain (CMC) is a Markov chain $X = (X_n)_{n \in \mathbb{N}}$ that is adapted to $\mathbb{F}$:
$$P(X_n \in \mathbb{F}_n) = 1 \quad \text{for all } n \in \mathbb{N}.$$

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AofA context: Consider an sequential algorithm that transforms a sequence $(\eta_n)_{n\in\mathbb{N}}$ of input values into a sequence $(x_n)_{n\in\mathbb{N}}$ of combinatorial objects such that

$$x_n \in \mathbb{F}_n, \quad x_{n+1} \text{ depends on } x_n \text{ and } \eta_{n+1} \text{ only.}$$

For i.i.d. input this algorithm generates a CMC.

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More assumptions:

- weak connectedness – there is a path from the root to each node,

$$\forall\, x \in \mathbb{F} \;\exists\, y_i \in \mathbb{F}_i,\; i = 1, \ldots, n := \phi(x) :$$
$$y_1 = e,\; y_n = x,\; (y_{i-1}, y_i) \in E(\mathbb{F}) \text{ for } i = 2, \ldots, n$$

- the transition probabilities $p(x, y) = P(X_{n+1} = y | X_n = x)$ are adapted to this structure,

$$p(x, y) > 0 \iff x \hookrightarrow y,$$

Example 1: Random walk, urns, and special numbers

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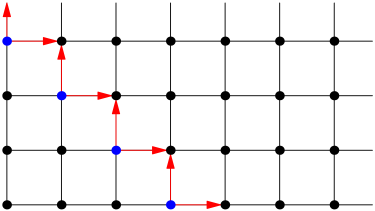

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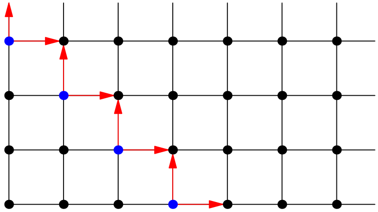

CMC	$p\big((i,j),(i+1,j)\big)$	$\mathcal{L}(X_n)$
north-east random walk	param. $\theta \in (0,1)$	$\binom{i+j}{i}\,\theta^i(1-\theta)^j$
"	$\theta = 1/2$	$\sim$ binomial coefficients
record walk	$1/(i+j+2)$	$\sim$ Stirling(1) numbers
Friedman urn	$(j+1)/(i+j+2)$	$\sim$ Euler numbers
Pólya urn	$(i+1)/(i+j+2)$	uniform distribution

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$$\mathbb{F}_n \ni \lambda = (\lambda_1, \ldots, \lambda_k) \hookrightarrow \mu = (\mu_1, \ldots, \mu_l) \in \mathbb{F}_{n+1}$$

iff $k \leq l$ and $\lambda_j \leq \mu_j$.

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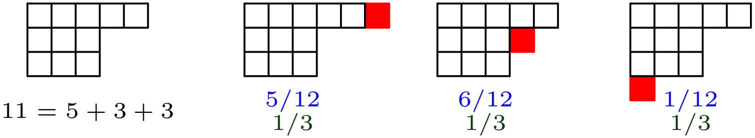

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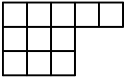
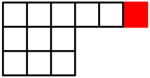
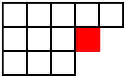


$11 = 5 + 3 + 3$



$5/12$

$1/3$



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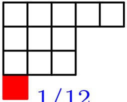

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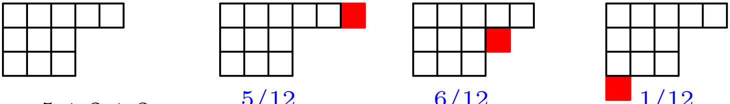

$11 = 5 + 3 + 3$ — $5/12$, $1/3$ — $6/12$, $1/3$ — $1/12$, $1/3$

- Thoma: $p(\lambda, \mu) = H(\lambda)/H(\mu)$, hook length formula.
- The restaurant (or cycle) chain:
  - If $\mu_j = \lambda_j + 1$, $p(\lambda, \mu) = \lambda_j \cdot \#\{i \geq j : \lambda_i = \lambda_j\}/(n+1)$.
  - If $\mu = (\lambda_1, \ldots, \lambda_k, 1)$: $p(\lambda, \mu) = 1/(n+1)$

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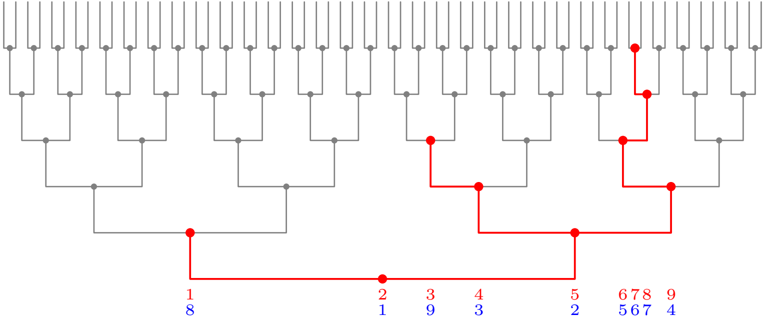

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Dirichlet problem	Brownian motion
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Connection to representation theory, random walks on groups:
Vershik, Kaimanovich, Kerov (book), Gnedin, Okounkov, Woess ...

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Then $\partial\mathbb{F} := \bar{\mathbb{F}} \setminus \mathbb{F}$ is the Martin boundary.

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defines another transition probability on $\mathbb{F}$.

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With $X_1^h = e$ and transitions p^h we obtain another CMC $X^h = (X_n^h)_{n \in \mathbb{N}}$, the *h*-transform of X.

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(reprDM)

The extensions $x \mapsto K(x, \alpha)$, $\alpha \in \partial\mathbb{F}$, are minimal harmonic, and all $h \in \mathbb{H}_+$ with $h(e) = 1$ are mixtures of these:

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(struct)

The conditional distribution $\mathcal{L}(X | X_\infty = \alpha)$ is equal to the distribution of the h-transform $\mathcal{L}(X^h)$ with $h = K(\cdot, \alpha)$ (in particular, this is the distribution of a Markov chain).

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$$E\big[K(x, X_{m-1})\big|X_m, \ldots, X_n\big] \leq K(x, X_m).$$

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Hence, for each $n \in \mathbb{N}$, $(Y_k^n, \mathcal{F}_k^n)_{k=1,\dots,n}$ with

$$Y_k^n := K(x, X_{n+1-k}), \quad \mathcal{F}_k^n = \sigma(X_{n+1-k}, \dots, X_n)$$

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is a supermartingale. For the number $U_n(a,b)$ of upcrossings of some interval $[a,b]$ this means

$$EU_n[a,b] \le \frac{1}{b-a}E(Y_n^n - a)^- = \frac{1}{b-a}\big(K(x,e)-a\big)^-.$$

It follows that $(K(x,X_n))_{n\in\mathbb{N}}$ converges almost surely.

Poisson boundary

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A measurable space $(E, \mathcal{B})$ together with a family ν_x, $x \in \mathbb{F}$, of probability measures on $(E, \mathcal{B})$ such that $\nu_x \ll \nu_e$ for all $x \in \mathbb{F}$ 'is' the Poisson boundary of $(X, \mathbb{F})$ if

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as Banach spaces, with the supremum norm on $\mathbb{H}_b$, via:

For each $\phi \in L^\infty$

(reprP) $$h(x) = \int_E \phi(y)\, \nu_x(dy), \quad x \in \mathbb{F},$$

is in $\mathbb{H}_b$, and for each $h \in \mathbb{H}_b$ there is a unique $\phi \in L^\infty$ such that this representation holds.

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For each $h \in \mathbb{H}_b^{\text{st}}$

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is a bounded martingale, hence $h(n,X_n) \to Y(h)$ a.s., with $Y(h)$ $\mathcal{T}$-mesurable. This leads to $L^\infty(\Omega,\mathcal{T},P \restriction \mathcal{T}) \cong \mathbb{H}_b^{\text{st}}$ via

(reprT) $$h(n,x) = E\big[Y(h)\big|X_n = x\big].$$

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- From the topological Martin boundary $\partial\mathbb{F}$ we obtain the measure-theoretic Poisson boundary via

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- The Martin boundary does not change when passing to an h-transform, the Poisson boundary may do so.

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- CMCs have the space-time property, i.e. $n = \phi(X_n)$, hence tail boundary and Poisson boundary coincide.

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Note: The Poisson boundary for the NE random walk is trivial.

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Boundary theory approach:

- Try to guess Φ,
- work out Ψ_n,
- prove $\Phi_n - \Psi_n \to 0$.

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(silhouette $\rightarrow$ metric silhouette).

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Régnier 1989, Rösler 1991,$\cdots$, Bindjeme and Fill 2012, Gr. 2012

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Proof uses the contraction method (Rösler, Rüschendorf, Neininger).

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Profiles (from Régnier to Jabbour-Hattab)

Count the external nodes of x at height k:

$$\Psi_k(x) := \#\{u \in \mathbb{V} : |u| = k, u \notin x, \bar{u} \in x\}.$$

Chauvin, Drmota, Jabbour-Hattab (2001) and Chauvin, Klein, Marckert, Roualt (2005) used the martingales

$$M_n(z) := \prod_{j=1}^{n-1} \frac{j+1}{j+2z} \sum_{k=1}^{\infty} \Psi_k(X_n)\, z^k, \; z \in \mathbb{C},$$

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BTA 'explains' the martingale and gives an interpretation of the limit, but does not (easily) lead to a new proof of the known results.

From AofA to boundaries

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In particular, X_∞ must be a function of η: For each $u \in \mathbb{V}$, let

$$\tau_u := \inf\{n \in \mathbb{N} : X_n \ni u\},$$

be the entrance time of the node and let

$$\eta_{(i:0)} := 0 < \eta_{(i:1)} < \cdots < \eta_{(i:i)} < \eta_{(i:i+1)} := 1$$

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Conclusion: Add 'algorithmic' to the above structural elements.

Also: A representation may lead to convergence in stronger topologies.

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Note: The first one is local, the others are global.

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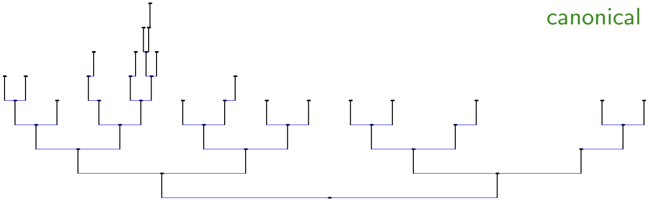

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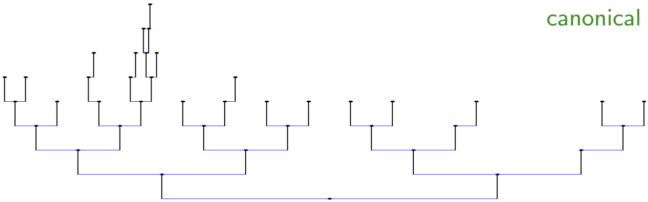

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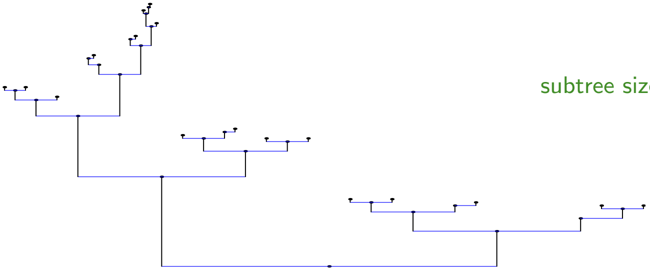

subtree size metric

Metric binary trees: Convergence

Any measure μ on $\{0,1\}^\infty$ defines a metric on $\mathbb{V}$ via

$$d_\infty(\bar{u}, u) = \mu(A_u) \quad \text{for all } u \in \mathbb{V},\ u \neq \emptyset,$$

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In view of $X_n \uparrow \mathbb{V}$ we can work with

$$(x_n, d_n) \to (\mathbb{V}, d_\infty)$$

meaning that

$$\lim_{n\to\infty} d_n(u,v) = d_\infty(u,v) \quad \text{for all } u, v \in \mathbb{V}.$$

Metric binary trees: A phase transition

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Let $\rho_0 = 1.26107\cdots$ *be the smaller one of the two roots of the equation* $2e\log(\rho) = \rho$, $\rho > 0$.

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