Weighted random staircase tableaux

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Staircase tableaux

A *staircase tableau of size n* is a Young diagram of shape $(n, n-1, \ldots, 2, 1)$ whose boxes are filled according to the following rules:

1. Each box is either empty or contains one of the letters α, β, δ, or γ.
2. No box on the diagonal is empty.
3. All boxes in the same row and to the left of a β or a δ are empty.
4. All boxes in the same column and above an α or a γ are empty.

An example of a staircase tableau of size 8. Its *weight* is $\alpha^5\beta^2\delta^3\gamma^3$.

	α						γ
	β			α		γ	
		α			γ		
				δ			
	δ		α				
		δ					
	β						
α							

Why?

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We consider the simplified case $u = q = 1$, which yields the weight above.

In our case, α and γ play the same role, and so do β and δ.

It is thus enough to study α/β-staircase tableaux, i.e. staircase tableaux with only α and β.

Let $S_{n,\alpha,\beta}$ be a random α/β-staircase tableau of size n, with probability proportional to the weight. (α, β are positive real numbers.)

Example

$S_{n,1,1}$ is a uniformly random α/β-staircase tableau.

Example

$S_{n,2,2}$ is a uniformly random staircase tableau (with $\alpha, \beta, \gamma, \delta$).

The total weight is, with $a = \alpha^{-1}$, $b = \beta^{-1}$,

$$Z_n(\alpha, \beta) = \prod_{i=0}^{n-1} (\alpha + \beta + i\alpha\beta) = \alpha^n \beta^n (\alpha^{-1} + \beta^{-1})^{\overline{n}}$$
$$= \alpha^n \beta^n \frac{\Gamma(n + a + b)}{\Gamma(a + b)}.$$

In particular, the number of α/β-staircase tableaux is

$$Z_n(1, 1) = (n + 1)!$$

There are explicit bijections with:

- permutations of size $n+1$,
- permutation tableaux of length $n+1$,
- alternative tableaux of length n,
- tree-like tableaux of length $n+2$.

Parameters

Given a staircase tableau S, let $N_\alpha, N_\beta, N_\gamma, N_\delta$ be the numbers of symbols $\alpha, \beta, \gamma, \delta$ in S.

Let A and B be the numbers of α and β, respectively, on the diagonal, and consider the random variables $A_{n,\alpha,\beta} := A(S_{n,\alpha,\beta})$ and $B_{n,\alpha,\beta} := B(S_{n,\alpha,\beta})$.

$$n \le N_\alpha + N_\beta + N_\gamma + N_\delta \le 2n - 1.$$

$$A_{n,\alpha,\beta} + B_{n,\alpha,\beta} = n.$$

Theorem

$$\mathbb{P}(A_{n,\alpha,\beta} = k) = \frac{v_{a,b}(n,k)}{(a+b)^{\overline{n}}}$$

where the numbers $v_{a,b}$ satisfy the recurrence

$$v_{a,b}(n,k) = (k+a)v_{a,b}(n-1,k) + (n-k+b)v_{a,b}(n-1,k-1), \qquad n \geq 1,$$

with $v_{a,b}(0,0) = 1$ and $v_{a,b}(0,k) = 0$ for $k \neq 0$.

[Carlitz and Scoville (1974)]

Special cases yield the Eulerian numbers:

$$v_{1,0}(n,k) = \left\langle {n \atop k} \right\rangle, \quad v_{0,1}(n,k) = \left\langle {n \atop k-1} \right\rangle, \quad v_{1,1}(n,k) = \left\langle {n+1 \atop k} \right\rangle$$

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Friedman's urn! (Add ball of opposite colour. Start with (a, b).)

Theorem

$$\mathbb{E}(A_{n,\alpha,\beta}) = \frac{n(n+2b-1)}{2(n+a+b-1)}$$

and

$$\mathrm{Var}(A_{n,\alpha,\beta}) = \dots$$

Asymptotic normality as $n \to \infty$.

Local limit theorem.

Theorem

$$\left(N_\alpha, N_\beta\right) \stackrel{\mathrm{d}}{=} \left(\sum_{i=0}^{n-1} I_i, \sum_{i=0}^{n-1} J_i\right),$$

where the pairs (I_i, J_i) *are independent of each other with*

$$I_i \sim \mathrm{Be}\Big(1 - \frac{a}{a+b+i}\Big), \qquad J_i \sim \mathrm{Be}\Big(1 - \frac{b}{a+b+i}\Big) \tag{1}$$

and $I_i + J_i \in \{1, 2\}$.

Theorem

$$\mathbb{E}\, N_\alpha = n - a \log n + O(1)$$
$$\operatorname{Var} N_\alpha = a \log n + O(1),$$
$$\operatorname{Cov}(N_\alpha, N_\beta) = O(1).$$

Joint asymptotic normality as $n \to \infty$.

Local limit theorem.

Poisson approximation.

Theorem

The number of α/β*-staircase tableau of size* n *with parameters* $A = k$, $B = n - k$, $N_\alpha = r$ *and* $N_\beta = s$ *equals the number of permutations of* $[n+1]$ *with* k *ascents,* $n - k$ *descents,* $n + 1 - s$ *left records and* $n + 1 - r$ *right records.*

Theorem

The number of α/β-staircase tableau of size n with parameters $A = k$, $B = n - k$, $N_\alpha = r$ and $N_\beta = s$ equals the number of permutations of $[n+1]$ with k ascents, $n-k$ descents, $n+1-s$ left records and $n+1-r$ right records.

Bijective proof??

Theorem

Delete the first i rows and the first j columns of $S_{n,\alpha,\beta}$. The resulting subtableau has the distribution of $S_{n-i-j,\hat{\alpha},\hat{\beta}}$ with $\hat{\alpha} = \hat{\alpha}^{-1} = a + i$, $\hat{\beta} = \hat{\beta}^{-1} = b + j$. ($i, j \geq 0$, $i + j < n$)

Theorem

Let $i + j < n$. The probability that the non-diagonal box (i, j) contains α or β is,

$$\mathbb{P}\big(S_{n,\alpha,\beta}(i,j) = \alpha\big) = \frac{j - 1 + b}{(i + j + a + b - 1)(i + j + a + b - 2)},$$
$$\mathbb{P}\big(S_{n,\alpha,\beta}(i,j) = \beta\big) = \frac{i - 1 + a}{(i + j + a + b - 1)(i + j + a + b - 2)},$$

and thus

$$\mathbb{P}\big(S_{n,\alpha,\beta}(i,j) \neq 0\big) = \frac{1}{i + j + a + b - 1}.$$

Proofs by generating functions, adding a new column on the left.