Urn models with multiple drawings

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joint work with May-Ru Chen; Hosam Mahmoud
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Urn models

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Pólya-Eggenberger urns

Urn contains n white and m black balls.

- Every discrete time steps a ball is drawn at random: $p_{\text{white}} = \frac{n}{n+m}$, $p_{\text{black}} = \frac{m}{n+m}$.
- Color inspection: White - a white and b black balls are added/removed; Black - c white and d black balls are added/removed; $a, b, c, d \in \mathbb{Z}$.
- 2×2 ball replacement matrix

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

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Pólya-Eggenberger urns

1. **Tenable urns**
   The process of drawing and adding/removing balls can be continued **ad infinitum**. We start with $W_0 \in \mathbb{N}_0$ white and $B_0 \in \mathbb{N}_0$ black balls: configuration W_n, B_n after n draws?

2. **Diminishing urns**
   The process of drawing and adding/removing balls stops after a **finite** number of steps. We start with $n \in \mathbb{N}_0$ white and $m \in \mathbb{N}_0$ black balls, define so-called absorbing states $\mathcal{A}$: what is the probability of reaching a state $a \in \mathcal{A}$?

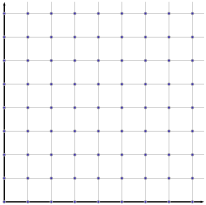
3. **Generalizations**
   More than two colors.

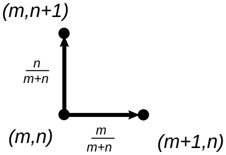
Pólya-Eggenberger urns

Pólya urn (tenable urns)

$M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Urn contains n white and m black balls:





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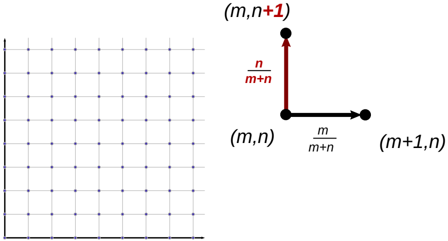

Pólya-Eggenberger urns

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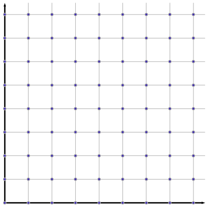

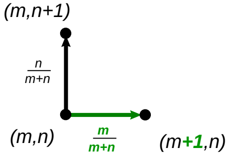

Pólya-Eggenberger urns

2000-today: several approaches - very interesting developments

Analytic Combinatorics (Symbolic methods, generating functions, etc.)

FLAJOLET, GABARRÓ AND PEKARI; BRENNAN AND PRODINGER; STADJE; DUMAS, FLAJOLET AND PUYHAUBERT; HWANG, K. AND PANHOLZER; FLAJOLET AND MORCRETTE; MAHMOUD AND MORCRETTE; MORCRETTE; ...

Probabilistic methods (stochastic processes, martingales)

KINGMAN^2; KINGMAN AND VOLKOV; MAHMOUD^x; PITTEL; JANSON^2; CHAUVIN, POUYANNE ET AL.^{3}; CHEN AND WEI; RENLUND; ...

Contraction method (for balanced urns)

NEININGER AND KNAPE; CHAUVIN, POUYANNE AND MAILLER; ...

...

An urn model $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is called **balanced**: $a + b = c + d = \sigma$.

Consequently: $T_n = W_n + B_n = T_0 + n \cdot \sigma$.

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Urn models

- Multiple drawings

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Urn models with multiple drawings

Previously: Urn contains w white and b black balls.

Draw at random a **single** ball: $p_{\text{white}} = \frac{w}{w+b}$, $p_{\text{black}} = \frac{b}{w+b}$.

New model: We draw $m \geqslant 1$ balls **without replacement**,

$$p_{\{k \text{ times white, } (m-k) \text{ times black }\}} = \frac{1}{(b+w)^{\underline{m}}} \binom{m}{k} w^{\underline{k}} b^{\underline{m-k}},$$

with $x^{\underline{s}} = x(x-1)\dots(x-s+1)$. Depending on the drawn multiset of white/black balls we add/remove balls.

Chen and Wei 2005: Generalized Pólya urn
Mahmoud 2008: Tenable balanced linear urns ($m = 2$).
Renlund 2010: Stochastic approximation for tenable urns.

Urn models with multiple drawings

Pólya urn: $M = \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}, c \in \mathbb{N}.$

Generalization: If we draw $\{W^k S^{m-k}\}$ we add $k \cdot c$ white and $(m-k) \cdot c$ black balls, with $c \in \mathbb{N}$. $(m+1) \times 2$-matrix

$$M = \begin{pmatrix} mc & 0 \\ (m-1)c & c \\ \dots & \dots \\ c & (m-1)c \\ 0 & mc \end{pmatrix}$$

Urn is balanced $T_n = W_n + B_n = nmc + T_0$.

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Friedman urn: $M = \begin{pmatrix} 0 & c \\ c & 0 \end{pmatrix}, c \in \mathbb{N}.$

Generalization: If we draw $\{W^k S^{m-k}\}$ we add $(m-k) \cdot c$ white and $k \cdot c$ black balls, with $c \in \mathbb{N}$. $(m+1) \times$ 2-matrix

$$M = \begin{pmatrix} 0 & mc \\ c & (m-1)c \\ \vdots & \vdots \\ (m-1)c & c \\ mc & 0 \end{pmatrix}$$

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Generalization: If we draw $\{W^k S^{m-k}\}$ we add $(m-k) \cdot c$ white and $k \cdot c$ black balls, with $c \in \mathbb{N}$. $(m+1) \times 2$-matrix

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Urn models

- Results

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Theorem (Chen and Wei 2005)

For the generalized Pólya urn the number of white balls W_n after n draws satisfies

$$\mathcal{W}_n = \frac{W_n}{T_n} \xrightarrow{\text{a. s.}} \mathcal{W}_\infty;$$

$\mathcal{W}_\infty$ *is absolutely continuous.*

Questions (Chen and Wei): (1) Is $\mathcal{W}_\infty$ **beta-distributed**? (2) Explicit results concerning W_n and $\mathcal{W}_\infty$;

Urn models with multiple drawings

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Urn models with multiple drawings

Theorem (Chen and K.)

The expectation and the variance of W_n are given by $\mathbb{E}(W_n) = \frac{W_0}{T_0}(nmc + T_0)$, *and* $\mathbb{V}(W_n) = \mathbb{E}(W_n^2) - \mathbb{E}(W_n)^2$ *via*

$$\mathbb{E}(W_n^2) = \frac{\binom{n-1+\lambda_1}{n}\binom{n-1+\lambda_2}{n}}{\binom{n-1+\frac{T_0}{mc}}{n}\binom{n-1+\frac{T_0-1}{mc}}{n}}\Bigg(W_0^2 + \frac{W_0c^2m}{T_0}\sum_{\ell=0}^{n-1}\frac{\ell+\frac{T_0-m}{mc}}{\ell+\frac{T_0-1}{mc}}\frac{\binom{\ell+\frac{T_0}{mc}}{\ell+1}\binom{\ell+\frac{T_0-1}{mc}}{\ell+1}}{\binom{\ell+\lambda_1}{\ell+1}\binom{\ell+\lambda_2}{\ell+1}}\Bigg),$$

with λ_1, λ_2 *given by*

$$\lambda_{1,2} = \frac{-\frac{1}{2} + mc + T_0 \pm \frac{1}{2}\sqrt{1+4mc(1+c)}}{mc}.$$

Urn models with multiple drawings

Theorem (CHEN AND K.)

For $s \geqslant 1$ the moment $\mathbb{E}(W_n^s)$ is given by

$$\mathbb{E}(W_n^s) = \left(\prod_{j=0}^{n-1} \alpha_{j,s}\right) \cdot \left(W_0^s + \sum_{\ell=0}^{n-1} \frac{\beta_{\ell,s}}{\prod_{j=0}^{\ell} \alpha_{j,s}}\right),$$

with $\alpha_{n,s}$, $\beta_{n,s}$ determined by

$$\alpha_{n,s} = \sum_{\ell=0}^{s} c^\ell \frac{\binom{s}{\ell}\binom{m}{\ell}}{\binom{T_n}{\ell}},$$

$$\beta_{n,s} = \sum_{i=2}^{s} \mathbb{E}(W_n^{s+1-i}) \sum_{\ell=i}^{s} \binom{s}{\ell} c^\ell \sum_{j=\ell+1-i}^{\ell} (-1)^{j+i-\ell-1} \frac{\left[{\ell \atop j}\right]\left\{{j \atop \ell+1-i}\right\}\binom{m}{j}}{\binom{T_n}{j}}.$$

Here $\left[{n \atop k}\right]$ and $\left\{{n \atop k}\right\}$ denote the Stirling numbers of the first and second kind, respectively.

Proof:

$$W_n \stackrel{(d)}{=} W_{n-1} + c \cdot \xi_n, \qquad \xi_n \stackrel{(d)}{=} \text{Hypergeometric}(m, W_{n-1}, B_{n-1});$$

in other words:

$$\mathbb{P}\big(\xi_n = k \mid \mathcal{F}_{n-1}\big) = \frac{\binom{W_{n-1}}{k}\binom{T_{n-1}-W_{n-1}}{m-k}}{\binom{T_{n-1}}{m}}.$$

Hence,

$$\mathbb{E}(W_n^s|\mathcal{F}_{n-1}) = W_{n-1}^s + \sum_{\ell=1}^{s}\binom{s}{\ell} W_{n-1}^{s-\ell} c^\ell \sum_{k=1}^{m} k^\ell \cdot \frac{\binom{W_{n-1}}{k}\binom{T_{n-1}-W_{n-1}}{m-k}}{\binom{T_{n-1}}{m}},$$

By Vandermonde's identity we obtain a recurrence

$$\mathbb{E}(W_n^s) = \alpha_{n-1,s} \cdot \mathbb{E}(W_{n-1}^s) + \beta_{n-1,s}, \qquad n, s \geq 1, \qquad (1)$$

which leads to the stated result.

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expand $k^\ell = \sum_{j=1}^{\ell} {\ell \brace j} k^{\underline{j}}$.

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which leads to the stated result.

Urn models with multiple drawings

Corollary (Chen and K.)

The limits $\lim_{n\to\infty} \mathbb{E}(W_n^s/n^s)$ *exist and can be recursively calculated:* $\lim_{n\to\infty} \frac{\mathbb{E}(W_n)}{n} = \frac{W_0 mc}{T_0}$, *and*

$$\lim_{n\to\infty} \frac{\mathbb{E}(W_n^2)}{n^2} = \frac{\Gamma(\frac{T_0}{mc})\Gamma(\frac{T_0-1}{mc})}{\Gamma(\lambda_1)\Gamma(\lambda_2)} \times \left(W_0^2 + \frac{W_0 c^2 m}{T_0} \sum_{\ell=0}^{\infty} \frac{\ell + \frac{T_0-m}{mc}}{\ell + \frac{T_0-1}{mc}} \frac{\binom{\ell+\frac{T_0}{mc}}{\ell+1}\binom{\ell+\frac{T_0-1}{mc}}{\ell+1}}{\binom{\ell+\lambda_1}{\ell+1}\binom{\ell+\lambda_2}{\ell+1}} \right).$$

Another model

(CHEN AND K.): draw $m \geqslant 1$ balls **with replacement**,

$$p_{\{k \text{ times white}, (m-k) \text{ times black}\}} = \frac{1}{(b+w)^m}\binom{m}{k} w^k b^{m-k}$$

Sampling scheme **influences** the (limiting) distribution.

Theorem

The expected value coincides: $\mathbb{E}(W_n) = \frac{W_0}{T_0}(nmc + T_0)$*; but for the second moment*

$$\mathbb{E}(W_n^2) = \frac{\binom{n-1+\mu_1}{n}\binom{n-1+\mu_2}{n}}{\binom{n-1+\frac{T_0}{mc}}{n}^2}\left(W_0^2 + \frac{W_0 c^2 m}{T_0}\sum_{\ell=0}^{n-1}\frac{\binom{\ell+\frac{T_0}{mc}}{\ell+1}^2}{\binom{\ell+\mu_1}{\ell+1}\binom{\ell+\mu_2}{\ell+1}}\right)$$

where the values μ_1, μ_2 *are given by* $\mu_{1,2} = \frac{T_0 + mc \pm c\sqrt{m}}{mc}$.

Urn models with multiple drawings

Friedman urn: $M = \begin{pmatrix} 0 & mc \\ \vdots & \vdots \\ mc & 0 \end{pmatrix}$ We obtain for both sampling schemes a unified result.

Theorem (Mahmoud, Panholzer and K.)

The number of white balls W_n after n draws satisfies

$$\frac{W_n}{n} \xrightarrow{(a.s.)} \frac{cm}{2}, \quad n \to \infty.$$

Furthermore,

$$\frac{W_n - \frac{1}{2}cmn - \frac{1}{2}T_0}{\sqrt{n}} \xrightarrow{(d)} \mathcal{N}\Big(0, \frac{1}{12}c^2m\Big),$$

with convergence of all moments.

Urn models

- Multiple drawings

Analytic combinatorics

Urn models

- Multiple drawings

Analytic combinatorics

General model:

$$M = \begin{pmatrix} a_0 & b_0 \\ a_1 & b_1 \\ \dots & \dots \\ a_{m-1} & b_{m-1} \\ a_m & b_m \end{pmatrix}.$$

Draw a multiset $\{W^k S^{m-k}\}$: we add a_k white and b_k black balls.

Analysis using Analytic combinatorics

Case $m = 1$: draw a **single** ball

- 2005 Flajolet, Gabarró and Pekari: **first order PDE** for balanced urns
- 2006 Dumas, Flajolet and Puyhaubert: **differential systems** for balanced urns
- 2010-2011 Flajolet and Morcrette: **AC** for **unbalanced** urns
- 2013 Morcrette: first order PDE for **unbalanced** urns!

Case $m = 2$:

- July 2010 Flajolet Second order PDE for Bernoulli-Laplace urn

Analysis using Analytic combinatorics

Balanced urns: $M = \begin{pmatrix} a_0 & b_0 \\ a_1 & b_1 \\ \dots & \dots \\ a_{m-1} & b_{m-1} \\ a_m & b_m \end{pmatrix}$, $\sigma = a_i + b_i$. Approach of DUMAS, FLAJOLET AND PUYHAUBERT: Study **urn histories** using differential operators:

∂_z: differential operator with respect to z; $\Theta_z = z \cdot \partial_z$.

Assume we have w white and b black balls $\Longrightarrow x^w y^b$:

$$y^{m-k} \partial_y^{m-k} x^k \partial_x^k (x^w y^b) = \frac{x^w y^b}{(b+w)^{\underline{m}}} \binom{m}{k} w^{\underline{k}} b^{\underline{m-k}},$$

$$\Theta_x^k \Theta_y^{m-k} (x^w y^b) = \frac{x^w y^b}{(b+w)^m} \binom{m}{k} w^k b^{m-k}.$$

Analysis using Analytic combinatorics

We introduce the differential operators

$$\mathcal{D}_M = \sum_{k=0}^{m} \binom{m}{k} x^{a_k+k} y^{b_k+m-k} x^k y^{m-k} \partial_x^k \partial_y^{m-k},$$

and

$$\mathcal{D}_R = \sum_{k=0}^{m} \binom{m}{k} x^{a_k} y^{b_k} \Theta_x^k \Theta_y^{m-k}.$$

Proposition

Starting with W_0 white and B_0 black balls the generating function of all urn histories $H(x, y; z) = \sum_{n \geqslant 0} \mathcal{D}^n x^{W_0} y^{B_0} \frac{z^n}{(n!)^m}$ satisfies

$$\mathcal{D} * H(x, y, z) = \frac{1}{z} \Theta_z^m * H(x, y, z),$$

with $\mathcal{D} = \mathcal{D}_M$ (without replacement) or $\mathcal{D} = \mathcal{D}_R$ (with replacement).

Further simplifications using $\Theta_x + \Theta_y = W_0 + B_0 + \sigma\Theta_z$.

Outlook

- Similar results for diminishing urn models; some second order PDEs are explicitly solvable (reduction to first order):

$M = \begin{pmatrix} -1 & 0 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}$. $M = \begin{pmatrix} -1 & 0 \\ -1 & -1 \\ 0 & -1 \end{pmatrix}$, $M = \begin{pmatrix} -1 & 0 \\ -1 & 0 \\ 1 & -2 \end{pmatrix}$, $M = \begin{pmatrix} -1 & 0 \\ 0 & -1 \\ 1 & -2 \end{pmatrix}$,
$M = \begin{pmatrix} -2 & 0 \\ -1 & 0 \\ 0 & -2 \end{pmatrix}$, $M = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \dots$

explicit results, moments, limit laws, ...

- Approach of **Morcrette** $\Rightarrow$ higher order PDE for unbalanced urns with multiple drawings...

Solvable higher order PDEs stemming from urn models

Thanks for your **attention**!

Thanks for your **attention**!