Sum of positions of records in random permutations: A precise analysis

Guy Louchard

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Introduction

The statistic *srec* is defined as the sum of positions of records in random permutations. The generating function (GF) of *srec* is given by

$$G(z) = \prod_{i=1}^{n}(z^i + i - 1), \tag{1}$$

and the probability generating function (PGF) is given by

$$Z(z) = \frac{\prod_{i=1}^{n}(z^i + i - 1)}{n!}. \tag{2}$$

This statistic has been the object of recent interest in the litterature. Let us mention
Kortchemski [6], where he obtains the GF (1) and also proves that

$$J_n(j) := [z^j]G(z) \sim e^{n\ln(n)y+\mathcal{O}(n)}, \text{ where } j = \frac{n(n+1)}{2}x, \quad x = 1-y^2, \tag{3}$$

$0 < y < 1$, (we assume that j is integer).

In [11], Prodinger analyzes the same statistic for a series of n random geometric variables (with weak and strong records). He obtains an asymptotic expansion of $\mathbb{E}(srec)$.
In [5], Knopfmacher and Mansour analyze *srec* in a random composition of an integer m.
In this paper, we analyze the central region $j = yn$ in Section 2 and we obtain, in Section 3 an asymptotic expansion generalizing (3). We use Cauchy's integral formula, the saddle point method and a technique developped in Arratia, Barbour and Tavare [1]. Section 4 provides the justification of the integration procedures used in the saddle point method. Section 5 concludes the paper.

The central region $j = yn$

From (2), we see that, as expected, $Z(1) = 1$. Moreover

$$Z'(z) = \sum_{i=1}^{n} \frac{iz^{i-1}}{z^i + i - 1} Z(z),$$

and

$$Z''(z) = \sum_{i=1}^{n} \frac{i[z^{i-2}i^2 - 2iz^{i-2} - z^{2i-2} + z^{i-2}]}{(z^i + i - 1)^2} Z(z) + \left(\sum_{i=1}^{n} \frac{iz^{i-1}}{z^i + i - 1} \right)^2 Z(z).$$

So

$$\mathbb{E}(srec) = Z'(1) = n,$$

and the variance is given by

$$\mathbb{V}(srec) = Z''(1) + n - n^2 = \sum_{i=1}^{n}(i-2) + n^2 + n - n^2 = \frac{n(n-1)}{2}.$$

Of course $Z(z)$ corresponds to a sum of independent *non* identically distributed random variables, but it is clear that the Lindeberg-Lévy conditions (see for instance, Feller [2]) are not satified here. The distribution is *not* asymptotically Gaussian. We tried to use the classical Saddle point method, but it appears that, at the Saddle point, the second derivative is of order n^2 but the third derivative is of order n^3!. So we couldn't apply this method.

Fortunately, we can use a technique developped in Arratia, Barbour and Tavare [1], sec. 4.2, which leads to an asymptotic distribution depending on the Dickman's function $\rho(x)$[1].
Equ. (2) can be written as

$$Z(z) = \prod_{i=1}^{n}(1 - \frac{1}{i} + \frac{z^i}{i}),$$

which corresponds to a sum of Bernoulli independent random variables, with parameter $1/i$. Let $V := srec/n$. Then

$$\mathbb{E}\left(e^{-sV}\right) = \exp\left(\sum_{1}^{n} \ln\left[1 + \frac{e^{-is/n} - 1}{i}\right]\right).$$

But $\frac{e^{-is/n}-1}{i} = \mathcal{O}(\frac{1}{n})$ uniformly in i, so

$$\mathbb{E}\left(e^{-sV}\right) = \exp\left(-\sum_{1}^{n} \frac{1 - e^{-is/n}}{i} + \mathcal{O}(\frac{1}{n})\right). \qquad (4)$$

[1] We are indebted to S.Janson for suggesting this use of Dickman's function

This is exactly the expression given in Arratia et al. [1] p.81, for $\theta = 1$. This leads to an asymptotic distribution given by $e^{-\gamma}\rho(v)$, where $\rho(v)$ is the Dickman's function. We obtain the folowing theorem

Theorem 2.1

The limiting distribution of $V := srec/n$*, in the central region, is given by*

$$e^{-\gamma}\rho(v),$$

and we conjecture the local limit theorem

$$\mathbb{P}(srec = j) \sim \frac{e^{-\gamma}\rho\left(\frac{j}{n}\right)}{n}.$$

Let us recall that $\rho(u)$ is the solution of

$$u\rho'(u) + \rho(u-1) = 0, \text{ for } u > 0,$$

and

$$\rho(u) = 0, \text{ for } u < 0, \rho(u) = 1, \text{ for } 0 \leq u \leq 1.$$

The first values of $\rho(u)$ are given by

$$\rho(u) = 1 - \ln(u), \text{ for } 1 \le u \le 2,$$

$$\rho(u) = 1 - [1 - \ln(1-u)]\ln(u) + Li_2(1-u) + \frac{\pi^2}{12} \text{ for } 2 \le u \le 3,$$

where

$$Li_2(z) := -\int_0^z \frac{\ln(1-t)}{t} dt.$$

Note that the function $dilog(z)$ often used in Computer algebra systems is given by

$$dilog(z) = Li_2(1-z).$$

From Thm 2.1,we derive

$$\mathbb{P}(srec = j) \sim \frac{e^{-\gamma}}{n}, j \leq n. \qquad (5)$$

This is easily checked in our case. Indeed, for $j \leq n$,

$$\mathbb{P}(srec = j) = \frac{1}{n!}[z^j]\prod_{i=1}^{j}[z^i + i - 1]\prod_{u=j}^{n-1} u,$$

and, if j is large, by (5),

$$[z^j]\prod_{i=1}^{j}[z^i + i - 1] \sim j!\frac{e^{-\gamma}}{j}.$$

So

$$\mathbb{P}(srec = j) \sim e^{-\gamma}\frac{1}{n!}(j-1)!\prod_{u=j}^{n-1} u = \frac{e^{-\gamma}}{n}.$$

For j large enough, $j \leq n$, $\mathbb{P}(srec = j)$ is constant.

We have made a numerical comparison of $\mathbb{P}(srec = j), n = 200, j = 1..3n$ with $\frac{e^{-\gamma}\rho(y)}{n}$. This is given in Figure 1 and is quite excellent.

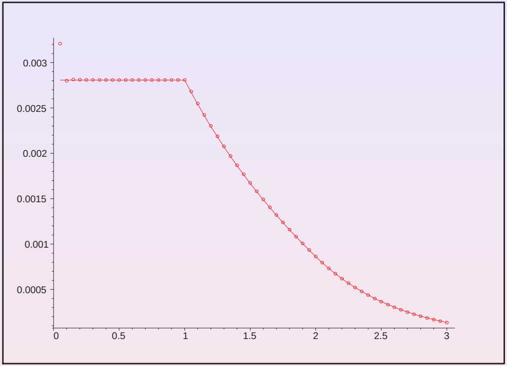

Figure 1: Comparison between $\mathbb{P}(srec = j), n = 200, j = 1..3n$ (circle) as function of y and the asymptotics $\frac{e^{-\gamma}\rho(y)}{n}$ (line)

The large deviation $j = \frac{n(n+1)}{2}(1 - y^2)$

The Saddle point method

By Cauchy's theorem,

$$\begin{aligned} J_n(j) &= \frac{1}{2\pi \mathbf{i}} \int_\Omega \frac{G(z)}{z^{j+1}} dz \\ &= \frac{1}{2\pi \mathbf{i}} \int_\Omega e^{S(z)} dz, \end{aligned} \tag{6}$$

where Ω is inside the analyticity domain of the integrand and encircles the origin and

$$S(z) = \sum_{i=1}^{n} \ln(z^i + i - 1) - \left(\frac{n(n+1)}{2}(1 - y^2) + 1 \right) \ln(z).$$

Set

$$S^{(i)} := \frac{d^i S}{dz^i}.$$

We will use the Saddle point method (for a good introduction to this method, see Flajolet and Sedgewick [3, ch. *VIII*]). First we must find the solution of

$$S^{(1)}(\tilde{z}) = 0$$

with smallest module. This leads to

$$\sum_{i=1}^{n} \frac{i\tilde{z}^i}{\tilde{z}^i + i - 1} - \left(\frac{n(n+1)}{2}(1-y^2)+1\right) = 0. \tag{7}$$

The left-hand side of this equation is an increasing function of $\tilde{z}$, so the solution $\tilde{z}$ is unique.

In some previous papers (see Louchard and Prodinger [9], [7], [10]), we simply tried $\tilde{z} = z^* + \varepsilon$ for some z^*, plug into (7), and expanded into ε. Here it appears that we cannot get this expansion. So we expand first (7) itself. But we must be careful.

There exists some $\tilde{i}$ such that $\tilde{z}^{\tilde{i}} = \tilde{i}$. Some numerical experiments suggest that $\tilde{i} = \mathcal{O}(n)$. So we set $\tilde{i} = \alpha n, 0 < \alpha < 1$ and we must now compute α.
We obtain $\tilde{z} = e^{\xi} > 1$, with

$$0 < \xi = \frac{L + \ln(\alpha)}{\alpha n} \ll 1,$$

where here and in the sequel, $L := \ln(n)$. Note that this leads to

$$\tilde{z}^n = \exp\left(\frac{L + \ln(\alpha)}{\alpha}\right) = n^{1/\alpha}\alpha^{1/\alpha}.$$

Using the classical splitting of the sum technique, (see for instance Greene and Knuth, [4]), equ. (7), assuming $\tilde{i}$ integer, leads to (we provide in the sequel only a few terms in the expansions, but Maple knows and uses more)

$$\Sigma_1 := \sum_{i=1}^{\tilde{i}} \frac{i\tilde{z}^i}{\tilde{z}^i + i - 1},$$

$$\Sigma_2 := \sum_{i=\tilde{i}+1}^{n} \frac{i\tilde{z}^i}{\tilde{z}^i + i - 1} - \left(\frac{n(n+1)}{2}(1 - y^2) + 1\right).$$

As

$$\frac{\tilde{z}^i - 1}{i} < \frac{\tilde{z}^{\tilde{i}} - 1}{\tilde{i}} < \frac{\tilde{z}^{\tilde{i}}}{\tilde{i}} = 1, i < \tilde{i},$$

we have

$$\Sigma_1 = \sum_{i=1}^{\tilde{i}} \frac{\tilde{z}^i}{1 + \frac{\tilde{z}^i - 1}{i}} = \sum_{i=1}^{\tilde{i}} \tilde{z}^i \left[1 - \frac{\tilde{z}^i - 1}{i} + \left(\frac{\tilde{z}^i - 1}{i} \right)^2 + \dots \right] \quad (8)$$

The first summation is immediate

$$\sum_{i=1}^{\tilde{i}} \tilde{z}^i = \frac{\tilde{z}^{\tilde{i}+1} - 1}{\tilde{z} - 1} - \frac{\tilde{z}}{\tilde{z} - 1} \sim \frac{\alpha^2 n^2}{L + \ln(\alpha)} + \mathcal{O}(n). \tag{9}$$

For the next summations, we use Euler-Mac Laurin summation formula. First of all, the correction (to first order) arising from replacing the summations in (8) by integrals is given by

$$\frac{1}{2} + \frac{1}{2} \frac{\tilde{i}}{1 + \frac{\tilde{i}-1}{\tilde{i}}} = \frac{1}{2} + \frac{1}{2} \frac{\alpha n}{2 - 1/(\alpha n)} \sim \frac{1}{4} \alpha n. \tag{10}$$

Next, we must compute integrals such as

$$\int_1^{\tilde{i}} \tilde{z}^v \left(\frac{\tilde{z}^v - 1}{v} \right)^k dv. \tag{11}$$

But we know that

$$\int_1^{\tilde{i}} \tilde{z}^v \left(\frac{\tilde{z}^v - 1}{v} \right) dv = \int_1^{\tilde{i}} e^{\xi v} \left(\frac{e^{\xi v} - 1}{v} \right) dv$$
$$= Ei(1, -2\xi) - Ei(1, -\xi) + Ei(1, -\tilde{i}\xi) - Ei(1, -2\tilde{i}\xi)$$
$$= Ei(1, -2\xi) - Ei(1, -\xi) + Ei(1, -(L + \ln(\alpha))) - Ei(1, -2(L + \ln(\alpha))),$$

where $Ei(1, x)$ is the exponential integral of index 1 and we use suitable extensions (with Cauchy principal values):

$$Ei(1, x) := \int_x^{\infty} \frac{e^{-y}}{y} dy.$$

Setting $L_1 := L + \ln(\alpha) \gg 1$, we have

$$\Re(Ei(1, -\xi)) = -\gamma - \ln(\xi) - \xi - \frac{\xi^2}{4} + \dots,$$
$$\Re(Ei(1, -L_1)) = e^{L_1} \left[-\frac{1}{L_1} - \frac{1}{L_1^2} + \dots \right] = \alpha n \left[-\frac{1}{L_1} - \frac{1}{L_1^2} + \dots \right].$$

We use similar expansions for terms like (11). This finally leads, with (10) and (9), to

$$\Sigma_1 = n^2 \left[\frac{47\alpha^2}{60L} + \frac{\alpha^2(-564\ln(\alpha) - 155)}{720L^2} + \mathcal{O}(L^{-3}) \right] + \mathcal{O}(n).$$

Note that we must use enough terms in (8) to obtain a sufficient precision.

Now we turn to Σ_2. As

$$\frac{i-1}{\tilde{z}^i} < \frac{\tilde{i}+1-1}{\tilde{z}^{\tilde{i}+1}} < \frac{\tilde{i}}{\tilde{z}^{\tilde{i}}} = 1, i > \tilde{i},$$

we have

$$\Sigma_2 = \sum_{i=\tilde{i}+1}^{n} \frac{i}{1+\frac{i-1}{\tilde{z}^i}} - \left(\frac{n(n+1)}{2}(1-y^2)+1 \right)$$
$$= \sum_{i=\tilde{i}+1}^{n} i \left[1 - \frac{i-1}{\tilde{z}^i} + \left(\frac{i-1}{\tilde{z}^i} \right)^2 + \ldots \right] - \left(\frac{n(n+1)}{2}(1-y^2)+1 \right).$$

After all sumations and substitutions such as

$$\tilde{z}^{in} = (n^{1/\alpha}\alpha^{1/\alpha})^i, \tilde{z}^{in\alpha} = (n\alpha)^i,$$

$$\Sigma_2 = n^2\left[\frac{1}{2} - \frac{\alpha^2}{2} - \frac{47\alpha^2}{60L} + \dots\right] + \mathcal{O}(n) + \frac{n^3}{n^{1/\alpha}}\left[\frac{\alpha}{\alpha^{1/\alpha}L} + \dots\right]$$
$$- \left(\frac{n(n+1)}{2}(1-y^2) + 1\right) + \dots$$

$$S'(\tilde{z}) = \Sigma_1 + \Sigma_2 = n^2\left[\frac{1}{2} - \frac{\alpha^2}{2} + \frac{\psi(\alpha, \ln(\alpha))}{L^2} + \dots - (1-y^2)/2\right]$$
$$+ \mathcal{O}(n) + \frac{n^3}{n^{1/\alpha}}\left[\frac{\alpha}{\alpha^{1/\alpha}L} + \dots\right] + \dots = 0, \qquad (12)$$

for some function $\psi(\alpha, \ln(\alpha))$. Putting the coefficient of n^2 to 0, and solving wrt α gives the solution

$$\alpha^* = y - \frac{3307y}{1800L^2} + \frac{10936249y}{2160000L^4} + \dots \qquad (13)$$

Now we must consider the other terms of (12). First we must compare n with $\frac{n^3}{n^{1/\alpha}}$.
If $\alpha > \frac{1}{2}$, $\quad n^{3-1/\alpha} > n$ and vice-versa. The most interesting case is the case $\alpha > \frac{1}{2}$ (the other one can be treated by similar methods). Note that there are also other terms in (12) of order $n^{k-(k-2)/\alpha}, k \geq 4$. These terms are greater than n if $\alpha > (k-2)/(k-1)$.
Returning to (12), we first compute $n^{1/\alpha^*} = n^{1/y}\varphi(y, L)$, with

$$\varphi(y, L) = e^{L(1/\alpha^* - 1/y)} = 1 + \frac{3307}{1800yL} + \dots \qquad (14)$$

So we obtain from (12) the term

$$\frac{n^3}{n^{1/y}\varphi(y, L)}\left[\frac{\alpha^*}{\alpha^{*1/\alpha^*}L} + \dots\right],$$

and with (13),

$$\frac{n^3}{n^{1/y}}\left[\frac{y}{y^{1/y}L} + \dots\right].$$

Now we set $\alpha = \alpha^* + \frac{Cn}{n^{1/y}}$, plug into (12) (ignoring the $\mathcal{O}(n)$ term), and expand. The n^2 term must theoretically be 0. Actually, it is given by a series of large powers of $1/L$ as we only use a finite number of terms in (13). Solving the coefficient of $\frac{n^3}{n^{1/y}}$ wrt C, we obtain

$$C^* = \frac{1}{Ly^{1/y}} + \dots$$

and

$$\alpha = \alpha^* + \frac{C^* n}{n^{1/y}} + \dots \tag{15}$$

The computation of $S(\tilde{z})$

Proceeding as above, we have

$$S(\tilde{z}) = \Sigma_3 + \Sigma_4$$

with

$$\begin{aligned}
\Sigma_3 &= \sum_{i=1}^{\tilde{i}} \ln(\tilde{z}^i + i - 1) \qquad (16) \\
&= \sum_{i=1}^{\tilde{i}} \ln(i) + \sum_{i=1}^{\tilde{i}} \ln\left(1 + \frac{\tilde{z}^i - 1}{i}\right) \\
&= \sum_{i=1}^{\tilde{i}} \ln(i) + \sum_{i=1}^{\tilde{i}} \left[\frac{\tilde{z}^i - 1}{i} - \frac{1}{2}\left(\frac{\tilde{z}^i - 1}{i}\right)^2 + \dots\right]
\end{aligned}$$

and substituting $\tilde{z} = e^{\xi}$,

$$\Sigma_3 = n\left[\alpha L + \alpha(\ln(\alpha) - 1) + \frac{31\alpha}{36L} + \dots\right].$$

Next,

$$\begin{aligned}
\Sigma_4 &= \sum_{i=\tilde{i}+1}^{n} \ln(\tilde{z}^i + i - 1) - \left(\frac{n(n+1)}{2}(1-y^2)+1\right)\ln(\tilde{z}) \\
&= \sum_{i=\tilde{i}+1}^{n} \ln(\tilde{z}^i) + \sum_{i=\tilde{i}+1}^{n} \ln\left(1+\frac{i-1}{\tilde{z}^i}\right) \\
&- \left(\frac{n(n+1)}{2}(1-y^2)+1\right)\ln(\tilde{z}) \\
&= \sum_{i=\tilde{i}+1}^{n} i\xi + \sum_{i=\tilde{i}+1}^{n}\left[\frac{i-1}{\tilde{z}^i} - \frac{1}{2}\left(\frac{i-1}{\tilde{z}^i}\right)^2 + \dots\right] \\
&- \left(\frac{n(n+1)}{2}(1-y^2)+1\right)\xi \\
&= nL\left[\frac{-\alpha^2+y^2}{2\alpha} - \frac{1}{2}\frac{(\alpha^2-y^2)\ln(\alpha)}{L\alpha} + \dots\right] \\
&+ \frac{n^2}{n^{1/\alpha}}\left[-\frac{(\alpha+L+\ln(\alpha))\alpha}{(L+\ln(\alpha))^2\alpha^{1/\alpha}} + \dots\right] + \dots,
\end{aligned}$$

and, using (15), (14),

$$S(\tilde{z}) = \Sigma_3 + \Sigma_4 = nyL + n\left[y(-1+\ln(y)) + \frac{6191y}{3600L} + \dots\right] + \frac{n^2}{n^{1/y}L}\left[-\frac{y}{y^{1/y}} + \dots\right] + \dots \quad (17)$$

The computation of $S''(\tilde{z})$

We have

$$S''(\tilde{z}) = \sum_{i=1}^{n} \left[\frac{\tilde{z}^i i^2}{\tilde{z}^2(\tilde{z}^i + i - 1)} - \frac{\tilde{z}^i i}{\tilde{z}^2(\tilde{z}^i + i - 1)} - \frac{\tilde{z}^{2i} i^2}{\tilde{z}^2(\tilde{z}^i + i - 1)^2} \right] + \left(\frac{n(n+1)}{2}(1 - y^2) + 1 \right) / \tilde{z}^2.$$

Proceeding as above and omitting the details, we have (here we only use the n^3 term)

$$S''(\tilde{z}) = n^3 \left[\frac{4y^3}{5L} - \frac{4y^3(-18 + 5\ln(y))}{25L^2} + \dots \right]. \qquad (18)$$

Similarly, we have

$$S'''(\tilde{z}) = \mathcal{O}\left(\frac{n^4}{L} \right),$$

$$S^{(4)}(\tilde{z}) = \mathcal{O}\left(\frac{n^5}{L} \right).$$

The final integration

Now we obtain

$$J_n(j) = \frac{1}{2\pi \mathbf{i}} \int_\Omega \exp\left[S(\tilde z) + S^{(2)}(\tilde z)(z - \tilde z)^2/2! + \sum_{l=3}^{\infty} S^{(l)}(\tilde z)(z - \tilde z)^l/l!\right] dz$$

(note carefully that the linear term vanishes). Set $z = \tilde z + \mathbf{i}\tau$. This gives

$$J_n(j) = \frac{1}{2\pi} \exp[S(\tilde z)] \int_{-\infty}^{\infty} \exp\left[S^{(2)}(\tilde z)(\mathbf{i}\tau)^2/2! + \sum_{l=3}^{\infty} S^{(l)}(\tilde z)(\mathbf{i}\tau)^l/l!\right] d\tau. \tag{19}$$

We can now compute (19), for instance by using the classical trick of setting

$$S^{(2)}(\tilde z)(\mathbf{i}\tau)^2/2! + \sum_{l=3}^{\infty} S^{(l)}(\tilde z)(\mathbf{i}\tau)^l/l! = -u^2/2,$$

computing τ as a truncated series in u which gives (by inversion)

$$\tau = \frac{\sqrt{L}}{\ldots^{3/2}} \left[u\tau_1 + u^2\tau_2 + u^3\tau_3 + u^4\tau_4 + \ldots\right],$$

with

$$\tau_1 = \sqrt{\frac{5}{9y^2}} + \dots$$

$$\tau_2 = \mathcal{O}\left(\frac{\mathbf{i}\sqrt{L}}{\sqrt{n}}\right),$$

$$\tau_3 = \mathcal{O}\left(\frac{L}{n}\right),$$

$$\tau_4 = \mathcal{O}\left(\frac{\mathbf{i}L^{3/2}}{n^{3/2}}\right).$$

Setting $d\tau = \frac{d\tau}{du}du$, expanding w.r.t. n and integrating on $[u = -\infty..\infty]$, we finally obtain

Theorem 3.1

The asymptotic value of $J_n(j) := [z^j]G(z)$, in the large deviation domain considered here, is given by

$$J_n(j) \sim \tilde{J}_n(j) = \frac{e^{S(\tilde{z})}}{\sqrt{2\pi S''(\tilde{z})}}$$

where $S(\tilde{z}), S''(\tilde{z})$ are given by (17) and (18).

In Figure 2, we give, for $n = 150$, a comparison between $\ln(J_n(j))$ (circle) and $\ln(\tilde{J}_n(j))$ (line), as a function of y. The fit is quite good, but when y is close to 1. But j is then small and our symptotics are no more very efficient (we are outside the large deviation range). We also show the first approximation (3): nLy (blue line) which is only efficient for very large n (i.e. when L is large, see the linear n term in (17).

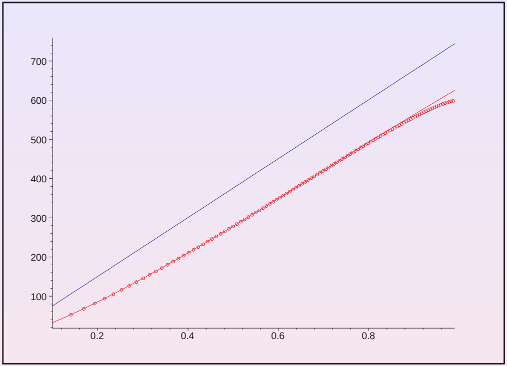

Figure 2: Comparison between between $\ln(J_n(j))$ (circle) and $\ln(\tilde{J}_n(j))$ (line), as a function of y, $n = 150$. Also it shows the first approximation (3): nLy (blue line)

The justification of the integration procedure is given in Section 4. Let us note that other large deviation regions can be analyzed by the same method. See for instance Louchard and Prodinger [8].

Justification of the integration procedures used in the large deviation saddle point method

This very technical proof is given in the full paper.

Conclusion.

Using the symbolic computer system Maple, we have obtained some asymptotic expressions for the sum of positions of records in random permutations in central and non-central regions. The Saddle point method proved again to be a powerful tool in our expansions computation.

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