Large Pólya Urns and Smoothing Equations

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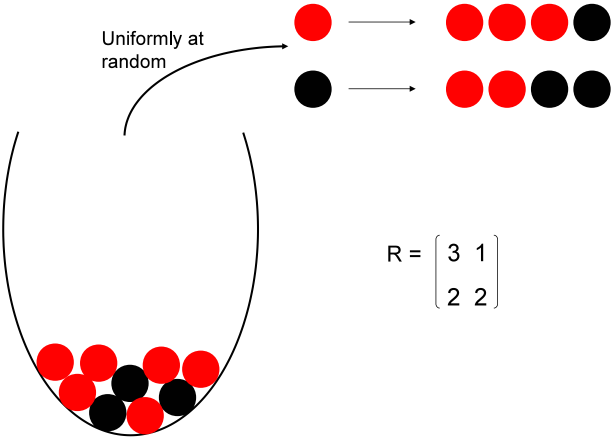
May 30, 2013

What is a Pólya urn?



$$R = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}$$

In general,

A two-colour Pólya urn is defined by:

- An initial composition:

$$U_0 = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

- A replacement matrix:

$$R = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

We assume:

- $a, b, c, d \geq 0$ (non extinction) and $bc \neq 0$,
- the urn is **balanced**, i.e. $a + b = c + d = S$

We denote by:

- m the second eigenvalue of R
- $\boxed{\sigma = \frac{m}{S}}$ the ratio of the two eigenvalues of R

Limit theorems

Composition vector:

$$U(n) = \begin{pmatrix} \#\text{ red balls at time } n \\ \#\text{ black balls at time } n \end{pmatrix}$$

Theorem ATHREYA, JANSON, ... :

- If $\sigma < \frac{1}{2}$

$$\frac{U(n) - nv_1}{\sqrt{n}} \xrightarrow[n\to+\infty]{(law)} \mathcal{N}(0, \Sigma^2).$$

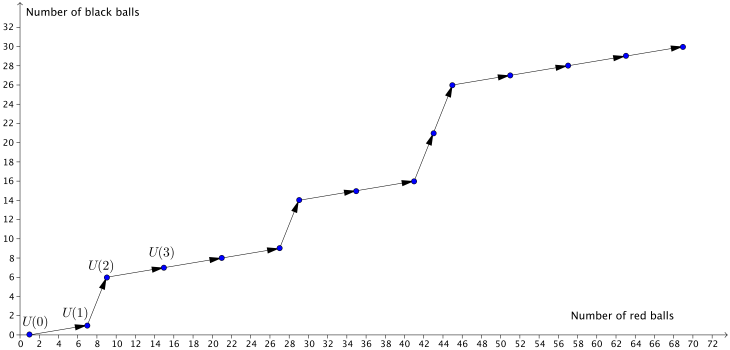
- If $\sigma > \frac{1}{2}$, then, a.s. and in all L^p $(p \geq 1)$,

$$U(n) = nv_1 + n^\sigma W^{DT} v_2 + o(n^\sigma).$$

where (v_1, v_2) is a (well chosen) basis of ${}^tR$, and (u_1, u_2) its dual basis.

Example of a trajectory

$$U(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad R = \begin{pmatrix} 6 & 1 \\ 2 & 5 \end{pmatrix}$$



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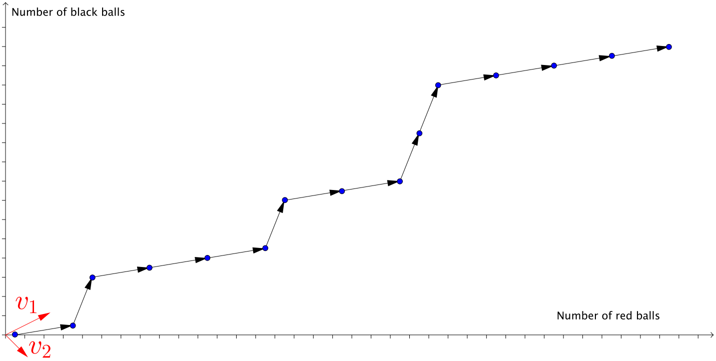

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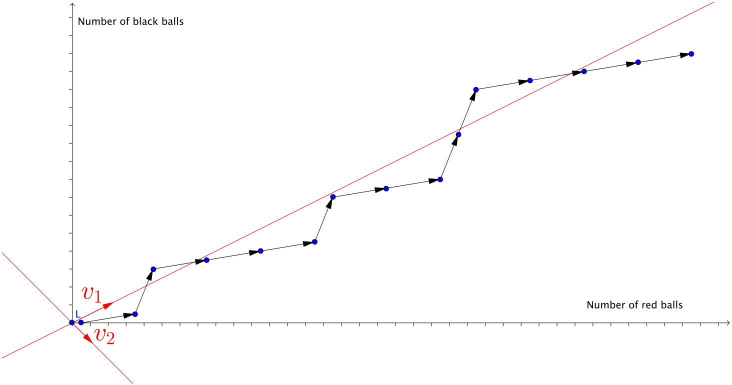


Embedding in continuous time

Each ball becomes a clock that rings after a random time of law $\mathcal{E}xp(1)$, **independently** from the others.

We denote by τ_n the date of the n^{th} ring,

$$(U(n))_{n\geq 0} = (U^{CT}(\tau_n))_{n\geq 0}, \text{ almost surely.}$$

Theorem ATHREYA, JANSON, ... :

If $\sigma > \frac{1}{2}$, asymptotically when n tends to $+\infty$, a.s. and in all L^p $(p \geq 1)$,

$$U^{CT}(t) = e^{St}\xi v_1(1 + o(1)) + e^{mt} W^{CT} v_2(1 + o(1)),$$

where ξ follows a Gamma$\left(\frac{\alpha+\beta}{S}\right)$ law.

Connexions

We are interested in the properties of W

$$W^{DT}_{(\alpha,\beta)} = \lim_{n\to+\infty} u_2\left(\frac{U_{(\alpha,\beta)}(n)}{n^\sigma}\right) \quad \text{and} \quad W^{CT}_{(\alpha,\beta)} = \lim_{t\to+\infty} u_2\left(\frac{U^{CT}_{(\alpha,\beta)}(t)}{e^{mt}}\right)$$

We know that (embedding in continuous time) :

$$W^{CT}_{(\alpha,\beta)} \overset{(law)}{=} \xi^\sigma W^{DT}_{(\alpha,\beta)} \quad \text{et} \quad W^{DT}_{(\alpha,\beta)} \overset{(law)}{=} \xi^{-\sigma} W^{CT}_{(\alpha,\beta)},$$

where ξ and $W^{DT}_{(\alpha,\beta)}$ are independent in the left identity.

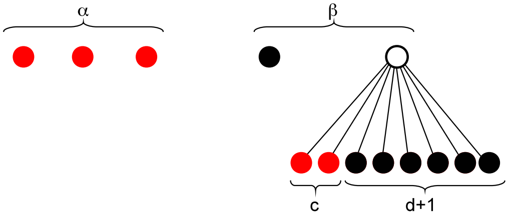
We also know CHAUVIN, ET AL '11 (among other results) that

- W^{CT} admits a density on $\mathbb{R}$
- we have an explicit but not very tractable formula for the Fourier transform of W^{CT}

Forest and urn in **disrcete time**

Composition of the urn = leaves in the forest.

Forest and urn



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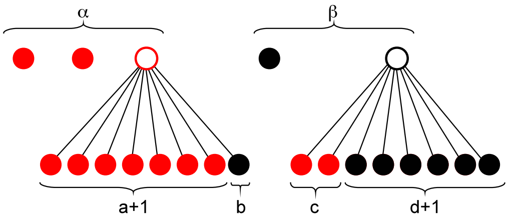

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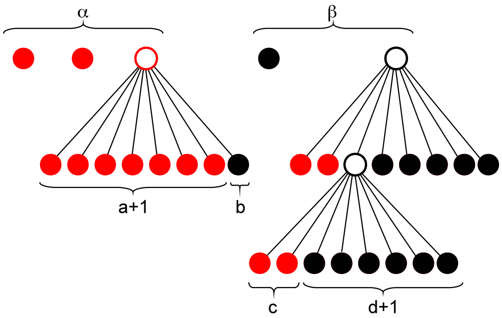
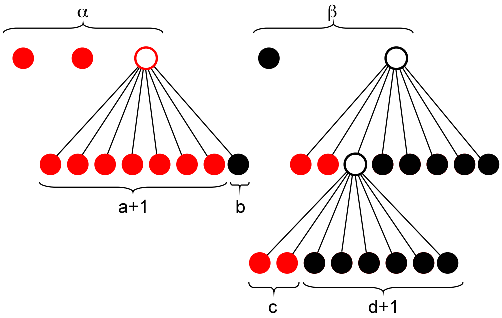


Composition of the urn = leaves in the forest.

$$U_{(\alpha,\beta)}(n) \overset{(law)}{=} \sum_{k=1}^{\alpha} U_{(1,0)}^{(k)}(T_k(n)) + \sum_{k=\alpha+1}^{\alpha+\beta} U_{(0,1)}^{(k)}(T_k(n)),$$

where $T_k(n)$ is the internal time in the k^{th} subtree.

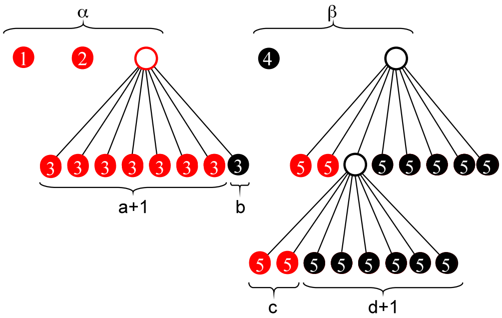
Forest and urn



$D_k(n)$ = number of leaves in the k^{th} tree of the forest at time n:

$$T_k(n) = \frac{D_k(n) - 1}{S} = \text{ internal “time” in the } k^{th} \text{ tree}$$

Forest and urn



$D_k(n)$ = number of leaves in the k^{th} tree of the forest at time n:

$(D_1(n), \dots, D_{\alpha+\beta}(n))$ is the composition vector of an urn with initial composition ${}^t(1, \dots, 1)$ and with replacement matrix $SI_{\alpha+\beta}$.

Remark:

$(D_1(n), \dots, D_{\alpha+\beta}(n))$ is the composition vector of an urn with initial composition ${}^t(1, \dots, 1)$ and with replacement matrix $SI_{\alpha+\beta}$.

Theorem ATHREYA **:**

$$\frac{1}{nS}(D_1(n), \dots, D_{\alpha+\beta}(n)) \xrightarrow[n\to+\infty]{a.s.} (V_1, \dots, V_{\alpha+\beta}),$$

where $(V_1, \dots, V_{\alpha+\beta})$ is Dirichlet$(\frac{1}{S}, \dots, \frac{1}{S})$-distributed.

$$U_{(\alpha,\beta)}(n) \overset{(law)}{=} \sum_{k=1}^{\alpha} U_{(1,0)}^{(k)}\left(\frac{D_k(n)-1}{S}\right) + \sum_{k=\alpha+1}^{\alpha+\beta} U_{(0,1)}^{(k)}\left(\frac{D_k(n)-1}{S}\right)$$

Forest and urn

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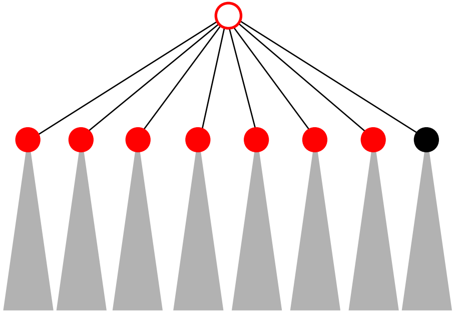
and since

$$W_{(\alpha,\beta)}^{DT} = \lim_{n\to+\infty} u_2\left(\frac{U_{(\alpha,\beta)}(n)}{n^{\sigma}}\right),$$

we have

$$\boxed{W_{(\alpha,\beta)} \overset{(law)}{=} \sum_{k=1}^{\alpha} V_k^{\sigma} W_{(1,0)}^{(k)} + \sum_{k=\alpha+1}^{\alpha+\beta} V_k^{\sigma} W_{(0,1)}^{(k)}}.$$

Let us study $W_{(1,0)}$ and $W_{(0,1)}$



$$W_{(1,0)} \overset{(law)}{=} \sum_{k=1}^{a+1} V_k^{\sigma} W_{(1,0)}^{(k)} + \sum_{k=a+2}^{S+1} V_k^{\sigma} W_{(0,1)}^{(k)}$$

$$W_{(0,1)} \overset{(law)}{=} \sum_{k=1}^{c} V_k^{\sigma} W_{(1,0)}^{(k)} + \sum_{k=c+1}^{S+1} V_k^{\sigma} W_{(0,1)}^{(k)}$$

Summary

- we reduced the study to $W_{(1,0)}$ and $W_{(0,1)}$
- $W_{(1,0)}$ and $W_{(0,1)}$ are solutions of a fixed point system:

$$\begin{cases} W_{(1,0)} \overset{(law)}{=} \sum_{k=1}^{a+1} V_k^\sigma W_{(1,0)}^{(k)} + \sum_{k=a+2}^{S+1} V_k^\sigma W_{(0,1)}^{(k)} \\ W_{(0,1)} \overset{(law)}{=} \sum_{k=1}^{c} V_k^\sigma W_{(1,0)}^{(k)} + \sum_{k=c+1}^{S+1} V_k^\sigma W_{(0,1)}^{(k)} \end{cases}$$

What information does the system give us?

Continuous time

We can use the same "tree" argument:

- we reduce the study to $W_{(1,0)}$ and $W_{(0,1)}$
- $W_{(1,0)}^{CT}$ and $W_{(0,1)}^{CT}$ are solutions of the following fixed point system:

$$\begin{cases} W_{(1,0)} \overset{(law)}{=} U^m \left(\sum_{k=1}^{a+1} W_{(1,0)}^{(k)} + \sum_{k=a+2}^{S+1} W_{(0,1)}^{(k)} \right) \\ W_{(0,1)} \overset{(law)}{=} U^m \left(\sum_{k=1}^{c} W_{(1,0)}^{(k)} + \sum_{k=c+1}^{S+1} W_{(0,1)}^{(k)} \right) \end{cases}$$

where U is uniformly distributed on $[0, 1]$.

What information does the system give us?

What information can we get from the systems?

Both in discrete and continuous time,

- Via the contraction method (i.e. Banach fixed point theorem in an appropriate complete metric space) KNAPE & NEININGER'S TALK,
  the solution of the system is unique at fixed mean
  n.b.: the system induces a contraction as soon as $\sigma = \frac{m}{S} > \frac{1}{2}$.
- **Existence of a density** : via Fourier transforms and Fourier inversion theorem (LIU's method for smoothing equations)
- Study of the moments : via the Carleman criterion and induction reasonning,
  $W_{(1,0)}$ **and** $W_{(0,1)}$ **are determined by their moments**
  and therefore, for all initial condition, $W_{(\alpha,\beta)}$ is determined by its moments

Moments

We already know (Chauvin et al. '11) that the radius of convergence of the Laplace series of W^{CT} is zero: for all $C > 0$, for large enough p,

$$C^p \leq \frac{\mathbb{E}|W^{CT}|^p}{p!}.$$

Theorem:

For all initial condition (α, β), the variables $W^{CT}_{(\alpha,\beta)}$ and $W^{DT}_{(\alpha,\beta)}$ are determined by their moments.

Idea of proof: Work on $W^{CT}_{(1,0)}$ and $W^{CT}_{(0,1)}$ and use the Carleman criterion:

"a variable which moments $(m_p)_{p\geq 1}$ verifies $\sum_{p\geq 1} m_{2p}^{-1/2p} = +\infty$ is determined by its moments"

By induction, there exists a constant $A > 0$

$$\frac{\mathbb{E}|W^{CT}|^p}{p!} \leq (A \ln p)^p.$$

Conclusion and perspectives

Tree structure of the urn permits to

- reduce the study to few initial conditions,
- caracterize the laws of $W_{(1,0)}$ and $W_{(0,1)}$ by a system of fixed point equations,
- prove the existence of a density, study the moments

Still a lot of work:

- what is the exact order of the moments?
- what are the tails of the law?
- generalize to d-colour urns

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Thanks for your attention!