The height of the Lyndon tree

Philippe Chassaing and Lucas Mercier

Université de Lorraine
IECL

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- Standard right factor of w:
  smallest $s_k(w)$, $2 \leq k \leq n$.

Lyndon tree

Split at the standard right factor, and iterate.

000110010111001

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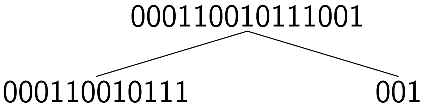

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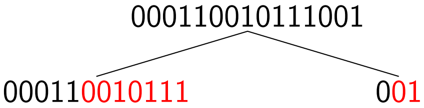

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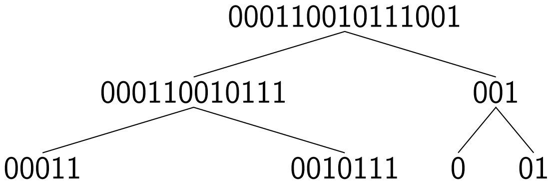

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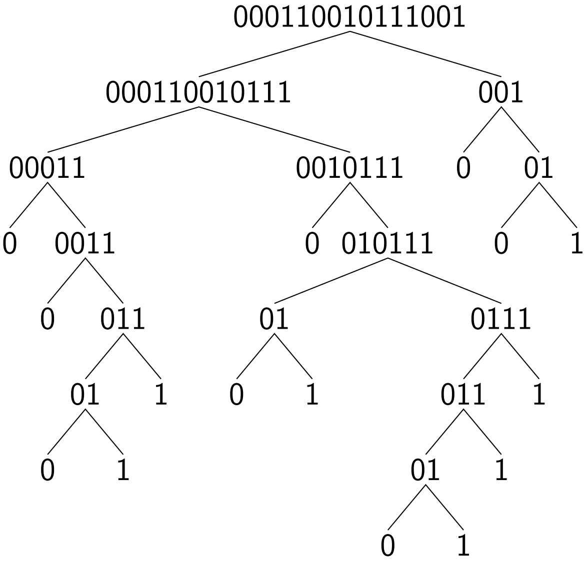

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No recursive structure.

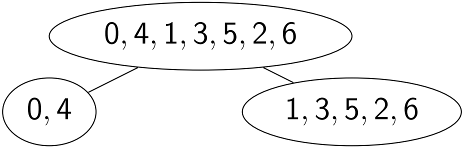
Binary search tree of a sequence

$$4, 1, 3, 5, 2, 6$$

Binary search tree of a sequence

$0, 4, 1, 3, 5, 2, 6$

Binary search tree of a sequence



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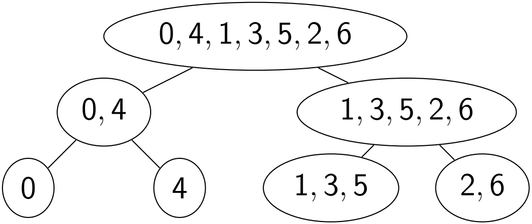

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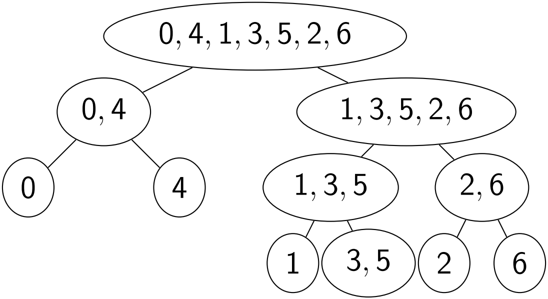

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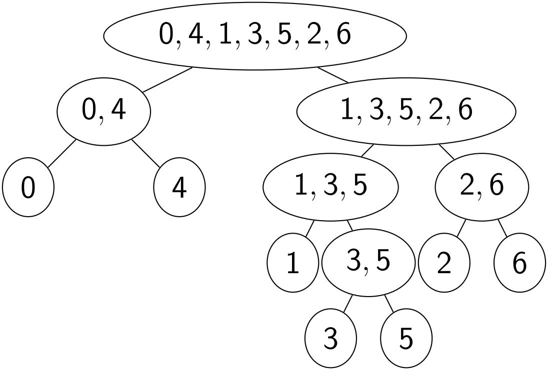

- Ordering the suffixes with the lexicographic order induces a permutation $\sigma(w)$ on $\{1, \ldots, n\}$.
  Example:$w = 0110$.
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- But $\sigma(w)$ is not uniform. Suffixes are not independent.

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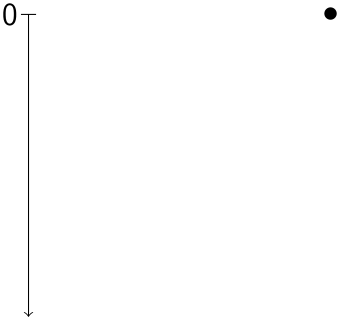
Echoes: if $s_k(w)$ starts with a l consecutive 0s, $s_{k+1}(w)$ starts with $l-1$ consecutive 0s.
At the bottom of the tree, suffixes are no longer independent...

Pseudo-height of a Yule tree.



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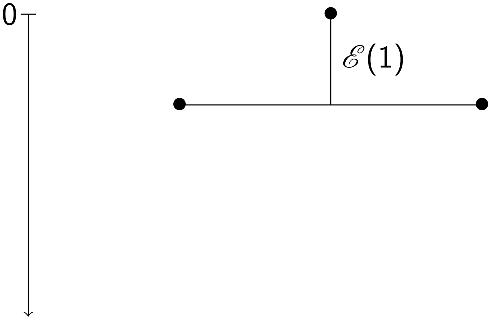

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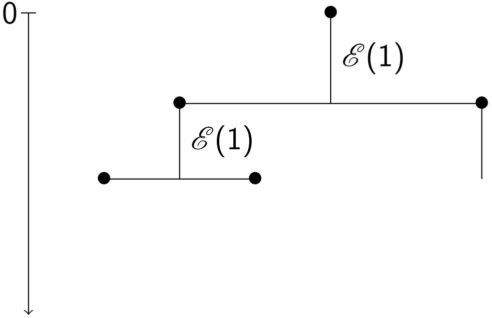

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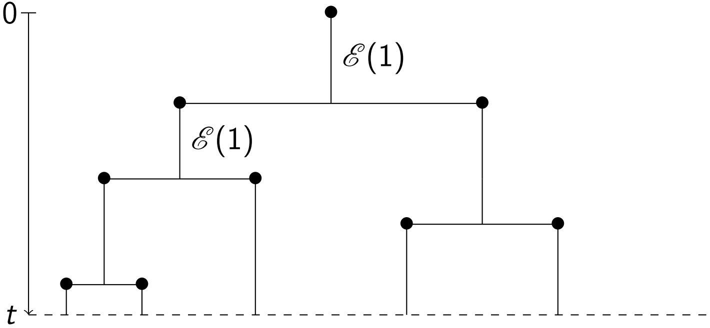

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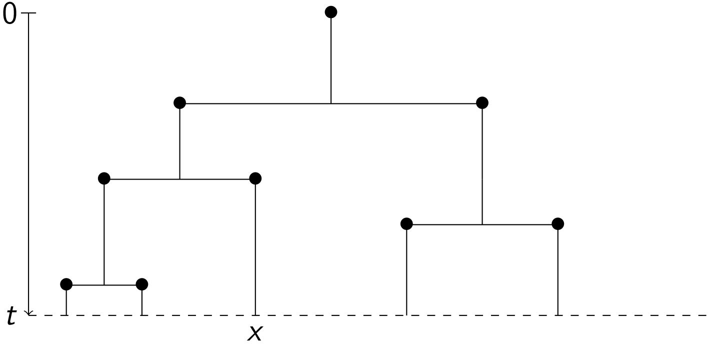

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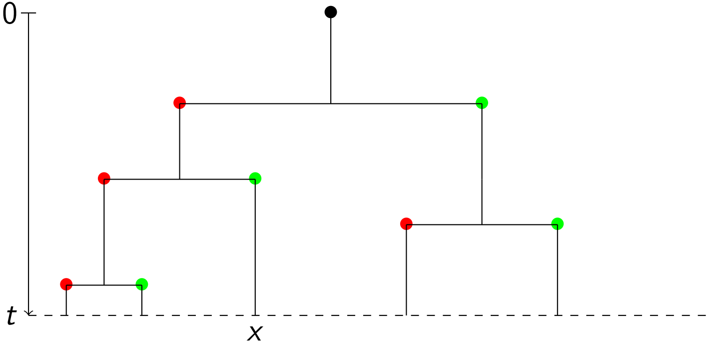

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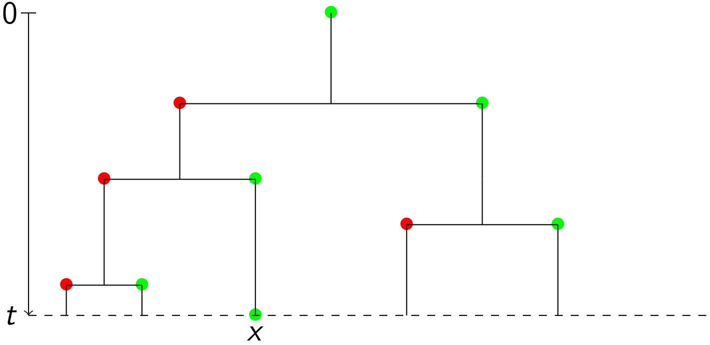

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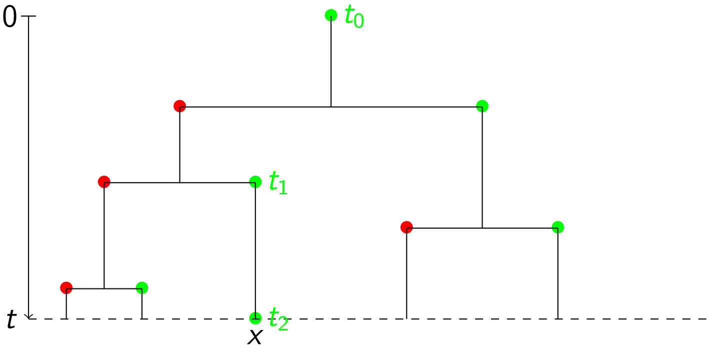

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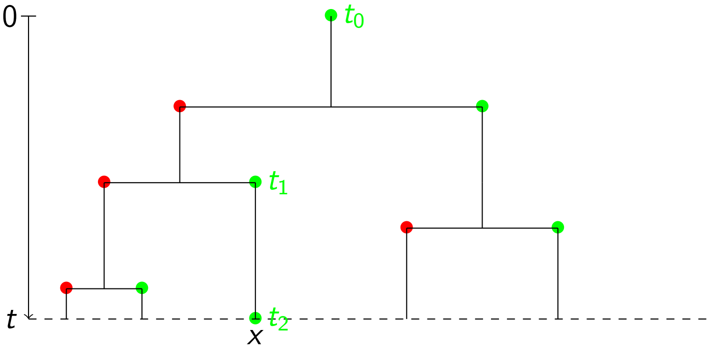

$$h_x = \begin{cases} \text{the depth of } x \text{ in the Yule tree} \\ +\sum_i \lfloor t_{i+1} - t_i \rfloor \\ +Geom(\frac{1}{2}) \end{cases}$$

Thanks for your attention!