Pólya urns via the contraction method

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joint work with **M. Knape**   **arXiv:1301.3404**

Pólya urns

Initial configuration: r_0 red balls, b_0 blue balls.

Replacement matrix:

	red	blue
red	a	b
blue	c	d

$a, d \in \mathbb{N}_0 \cup \{-1\}$, $b, c \in \mathbb{N}_0$

Methods

Main focus: # balls of each color, $\mathrm{n} \to \infty$

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Calculations with moment generating function

Calculations with moments

Martingale methods

Embedding into continuous time branching processes

Counting urn histories + analytic combinatorics

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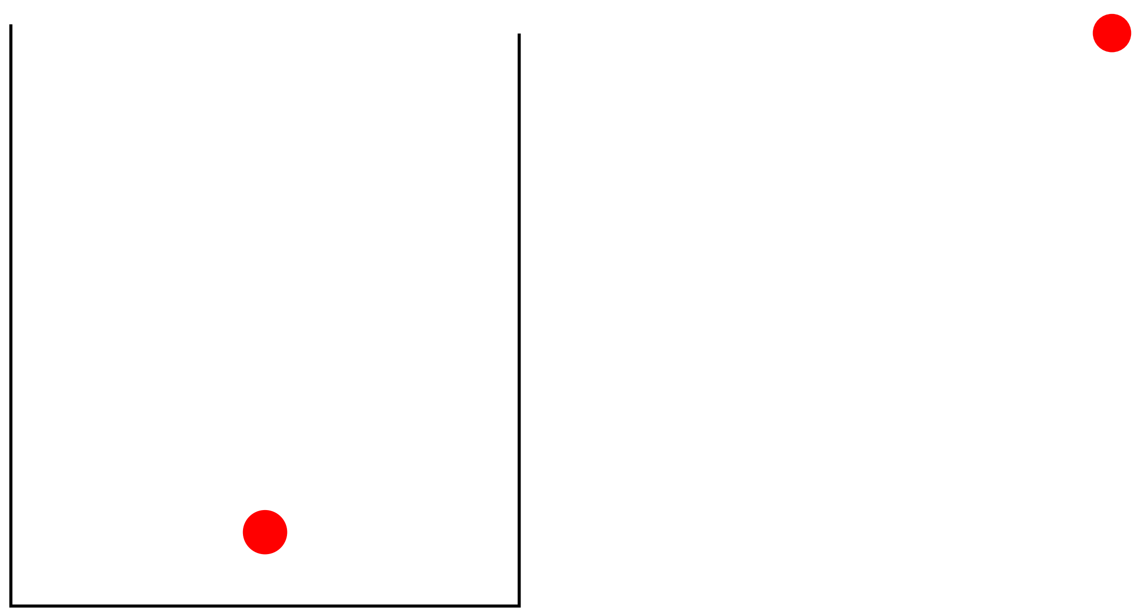
Here: **Recursive approach**

$a + b = c + d$: **“balanced urn”**

A discrete-time embedding



A discrete-time embedding

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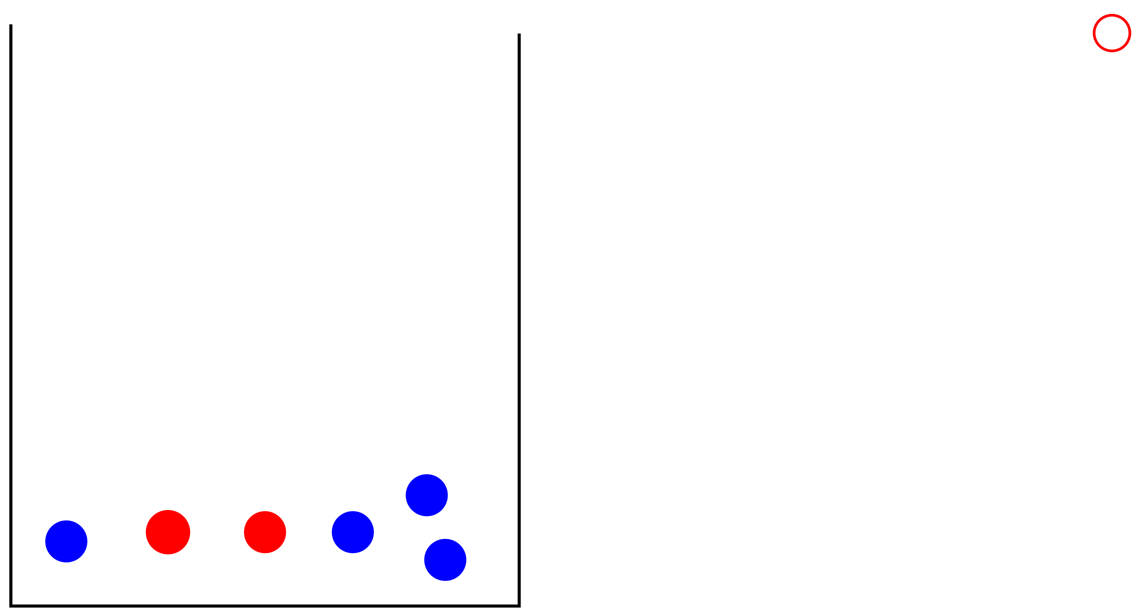

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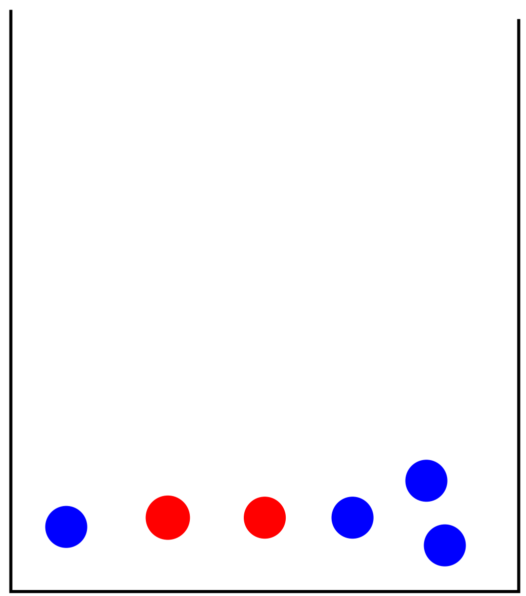

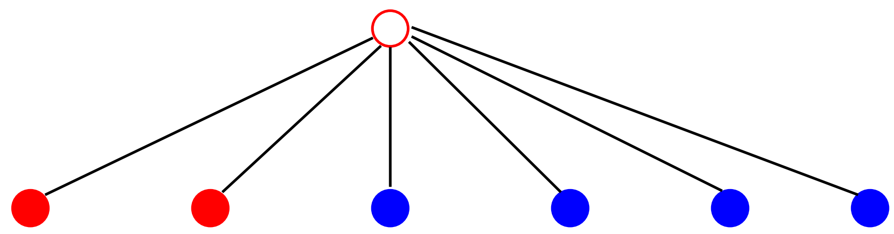

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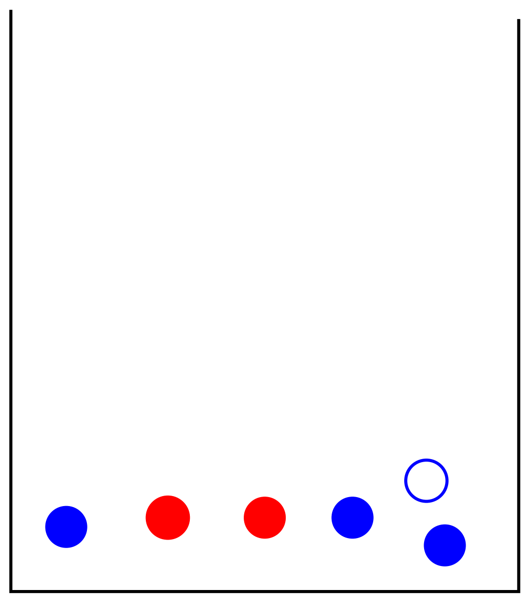

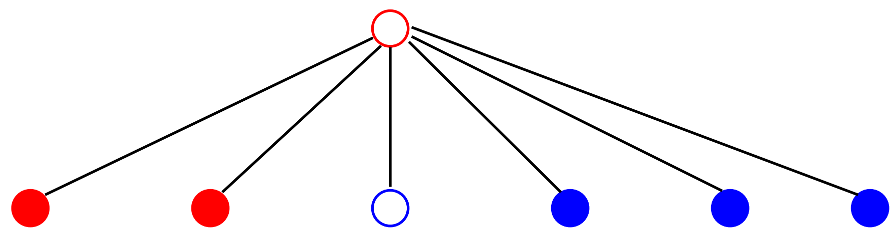

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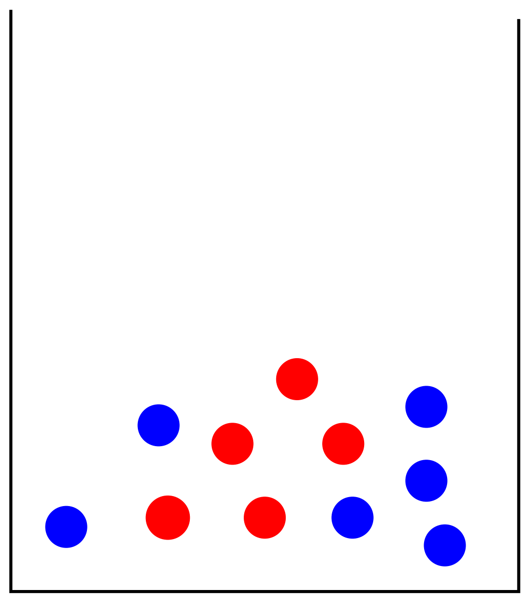

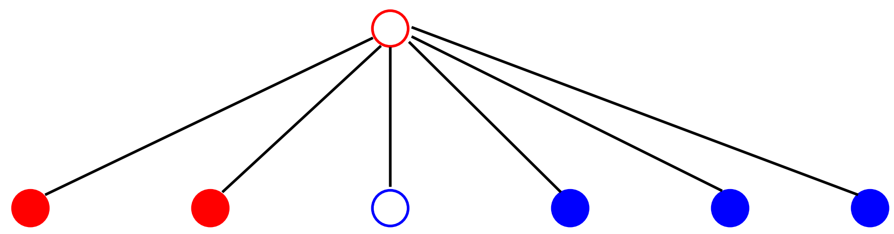

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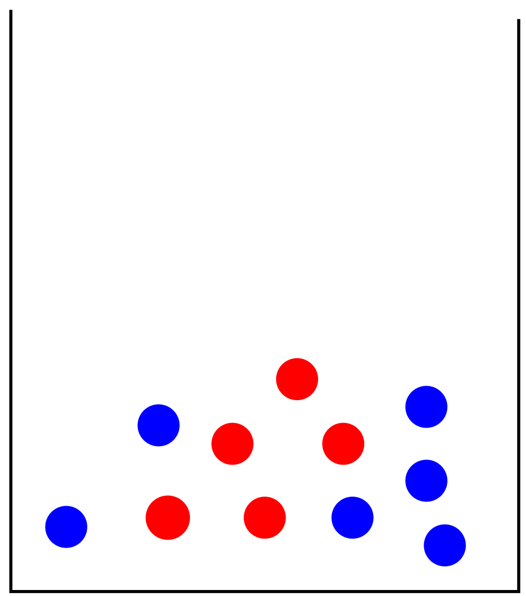

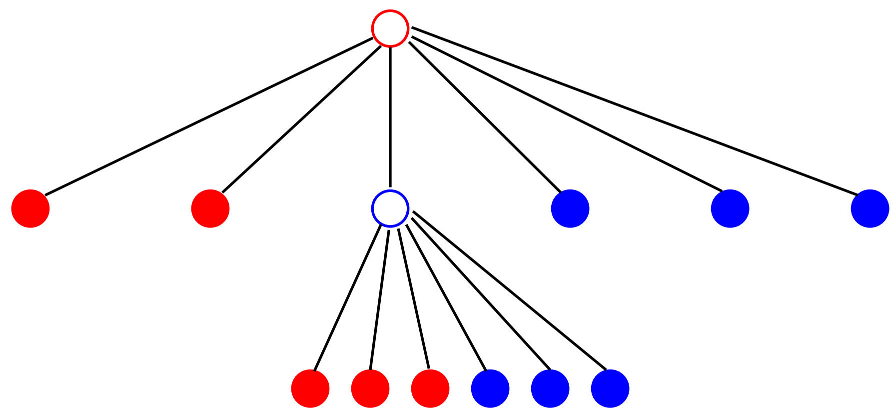

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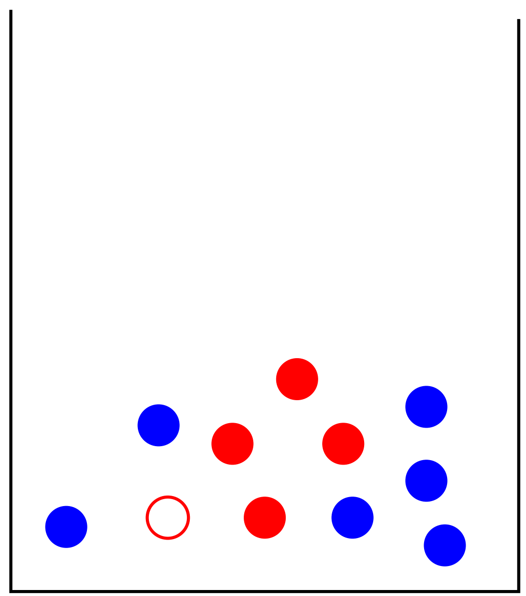

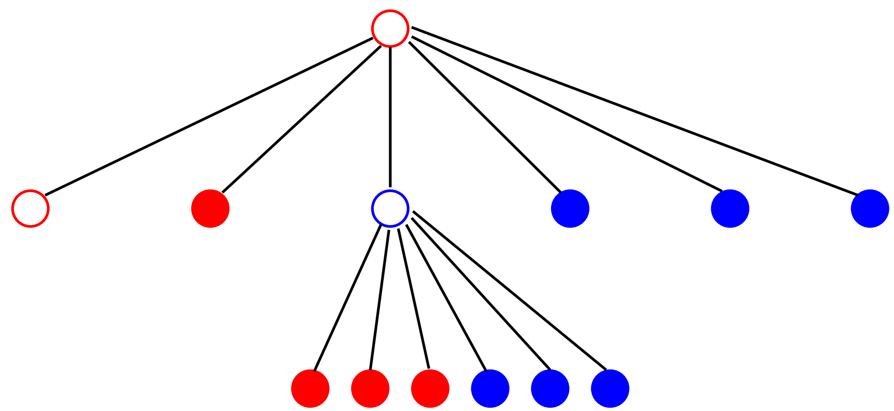

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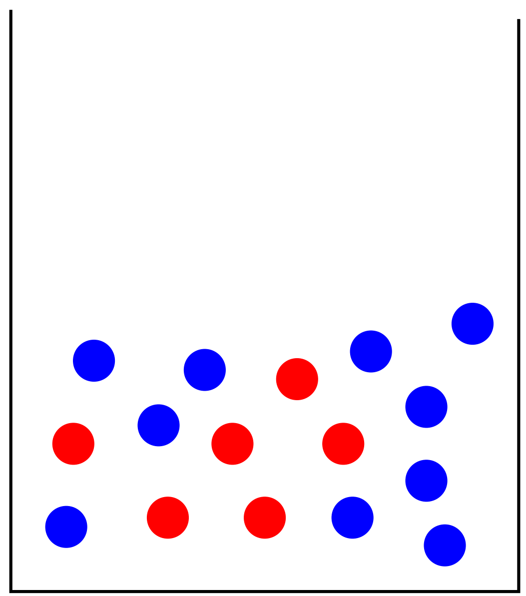

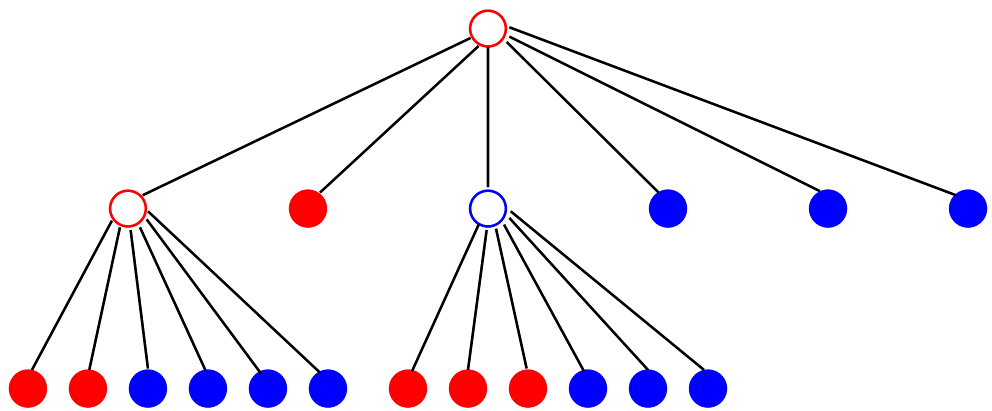

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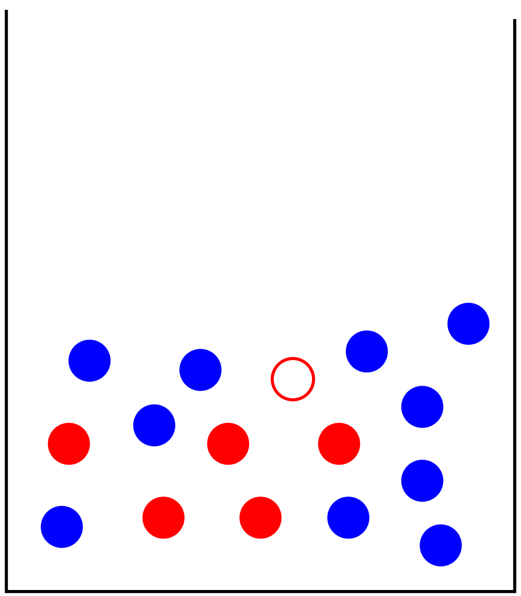

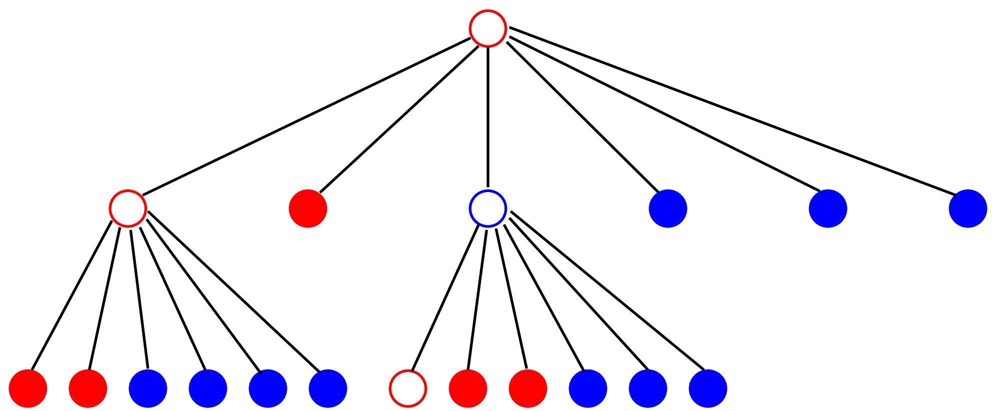

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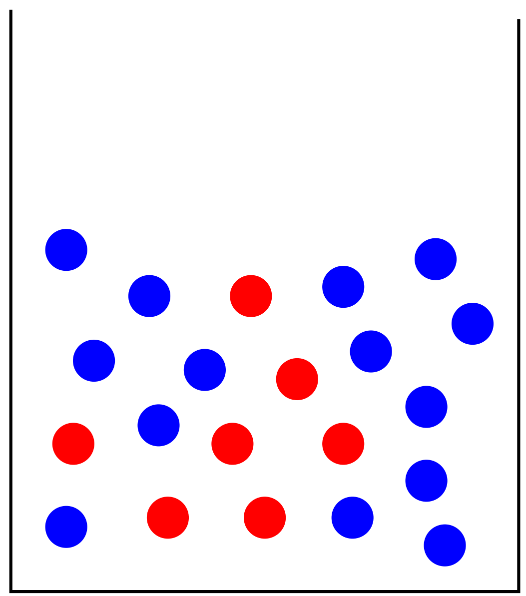
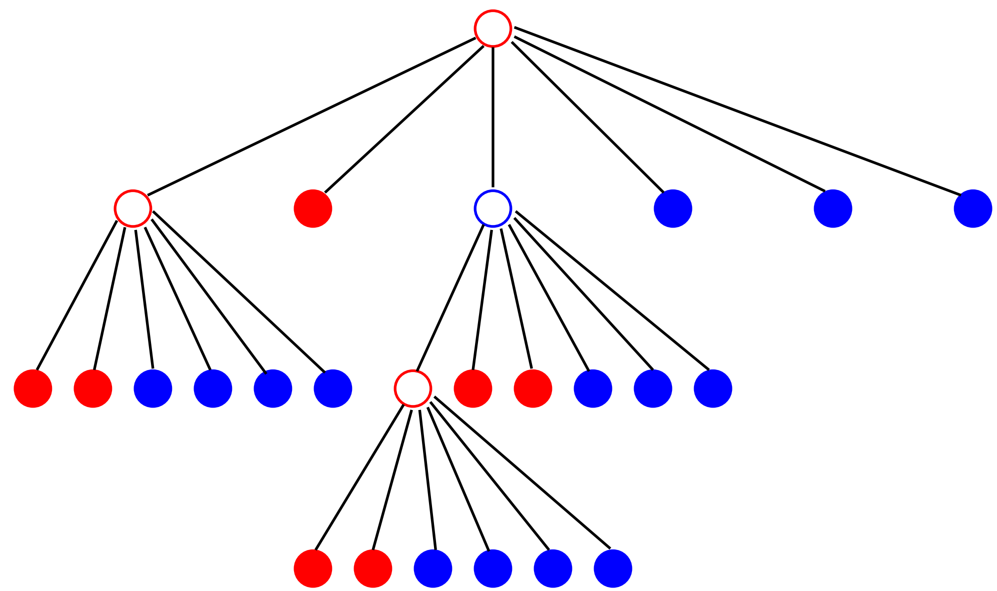
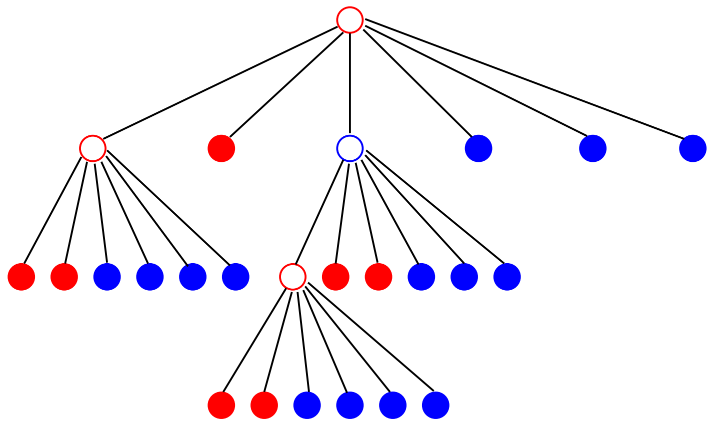




Embedding into trees



Balls are held in leaves of the tree.

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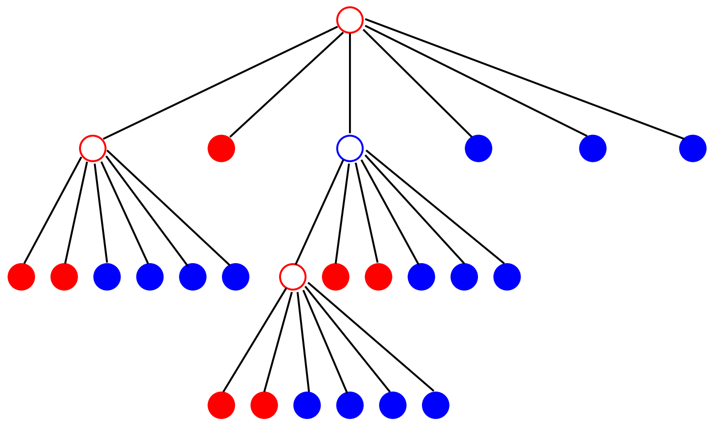

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Balanced urn: $a+b = c+d =: K-1$.

Branch degree: K.

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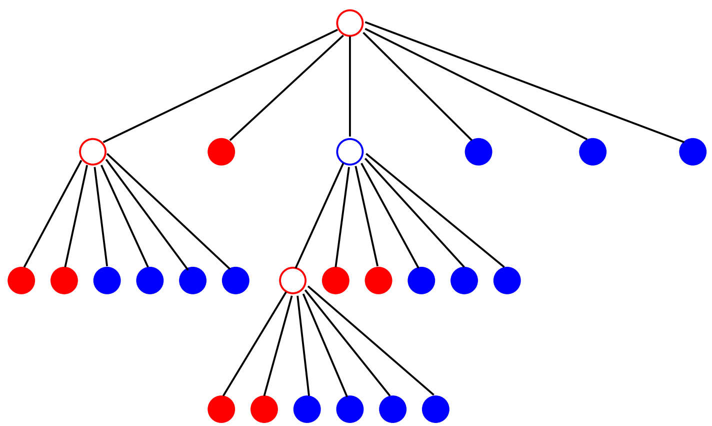

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$$\# \text{ red balls} = \sum_{j=1}^{K} \# \text{ red balls in } j\text{-th subtree.}$$

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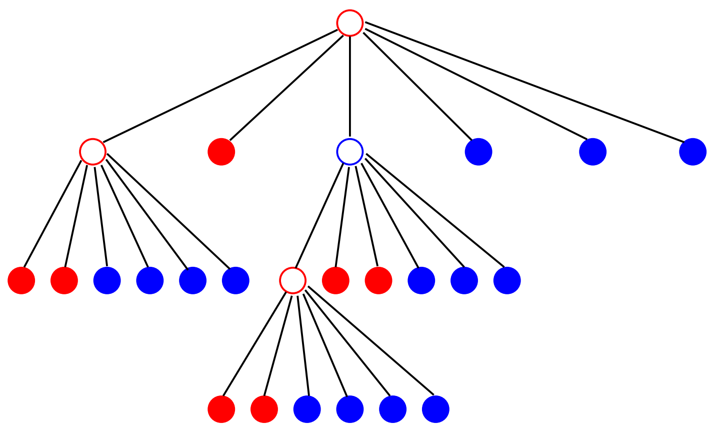

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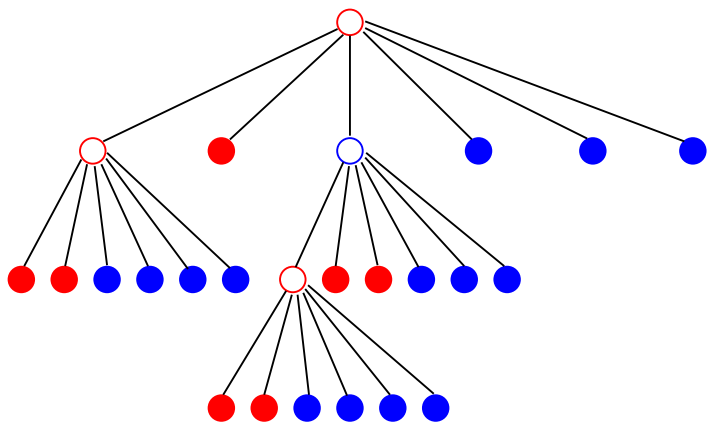
Subtrees rooted by red / blue (ghost) balls
behave differently.

General recurrence

$R_n^{(r)}$: # red balls when starting with red after n steps.
$R_n^{(b)}$: # red balls when starting with blue after n steps.



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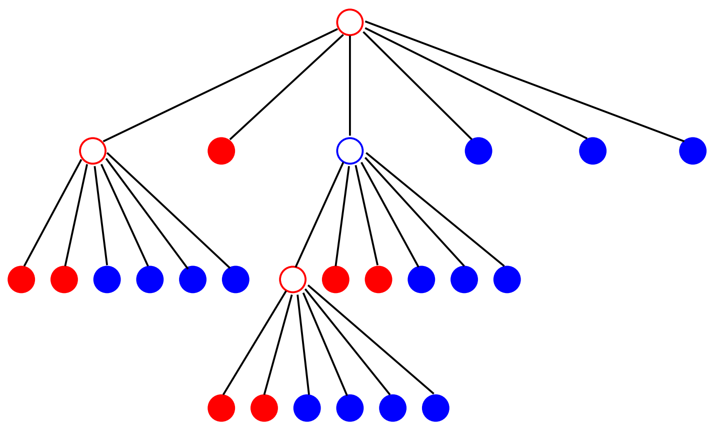
$I^{(n)} = (I_1, \ldots, I_K)$: **vector of # draws in each subtree.**



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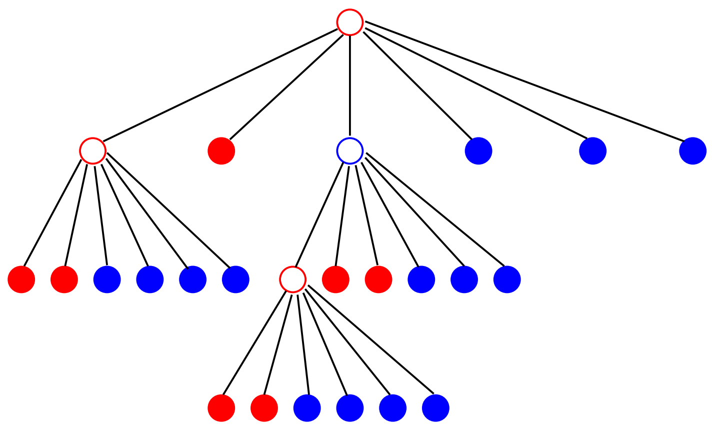

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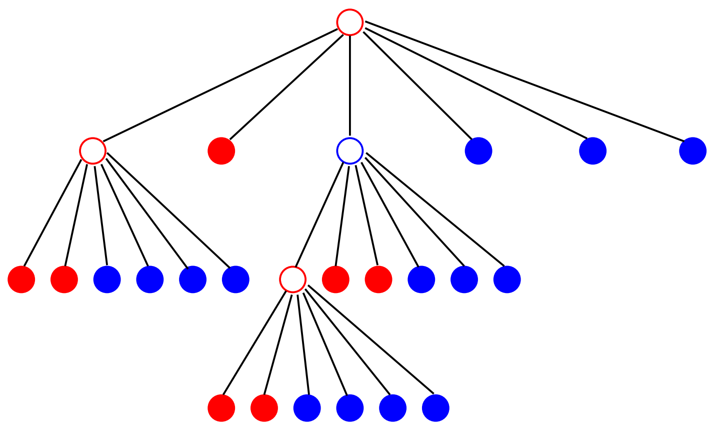

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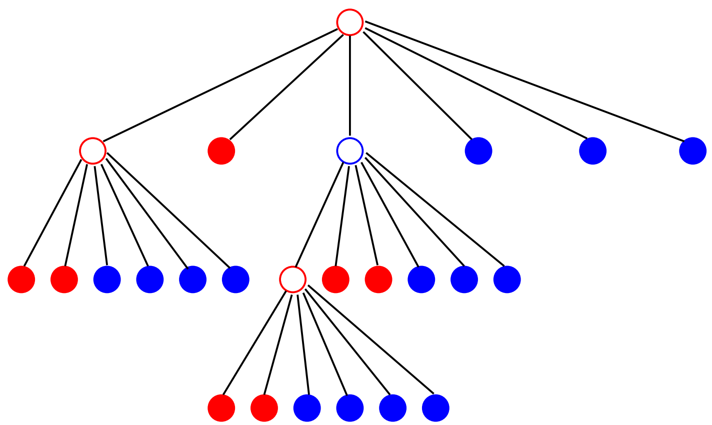

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Sizes of subtrees

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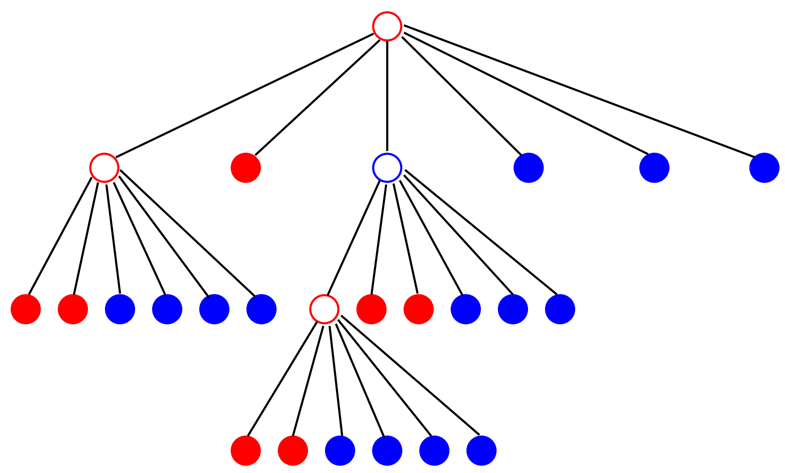

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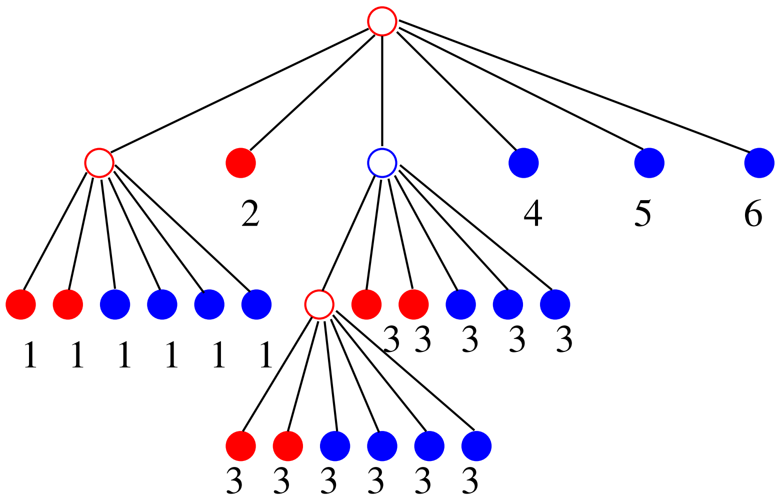
Label balls by their subtree.



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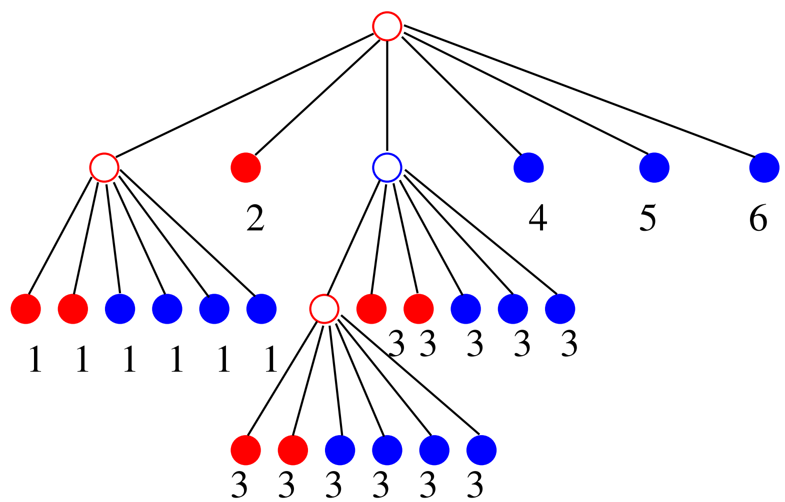

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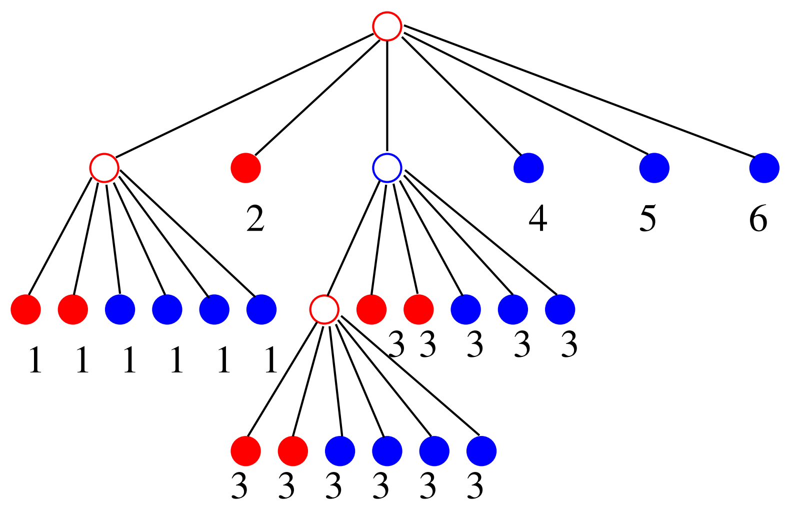

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Initial condition:

$(1, 1, \ldots, 1)$

Replacement matrix:

$\mathrm{diag}(K-1, K-1, \ldots, K-1)$

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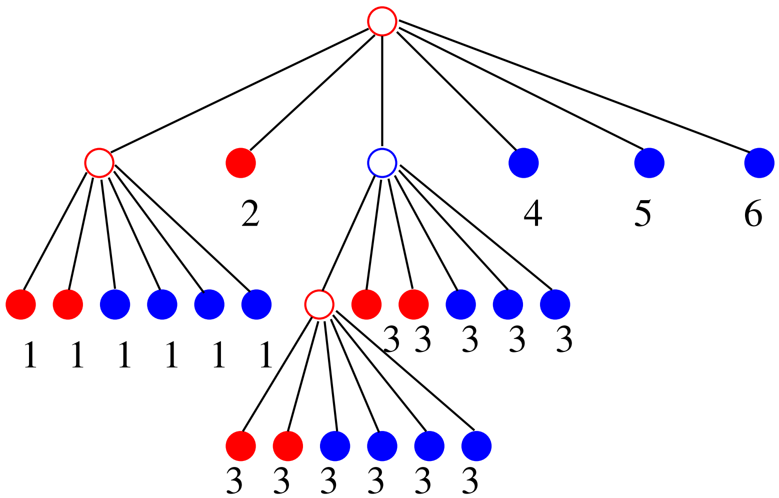

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$$\frac{1}{n} I^{(n)} \xrightarrow{\text{a.s.}} (D_1, \ldots, D_K)$$

With

$$(D_1, \ldots, D_K) \sim \text{Dirichlet}\left(\frac{1}{K-1}, \ldots, \frac{1}{K-1}\right).$$

Normalization

$$R_n^{(r)} \stackrel{d}{=} \sum_{j=1}^{a+1} R_{I_j}^{(r),j} + \sum_{j=a+2}^{K} R_{I_j}^{(b),j}$$

$$R_n^{(b)} \stackrel{d}{=} \sum_{j=1}^{c} R_{I_j}^{(r),j} + \sum_{j=c+1}^{K} R_{I_j}^{(b),j}$$

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Normalization

$$X_n^{(r)} := \frac{R_n^{(r)} - \mathbb{E}\, R_n^{(r)}}{n^\gamma L(n)}, \qquad X_n^{(b)} := \frac{R_n^{(b)} - \mathbb{E}\, R_n^{(b)}}{n^\gamma L(n)}$$

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Modified equations

$$X_n^{(r)} \stackrel{d}{=} \sum_{j=1}^{a+1} \left(\frac{I_j}{n}\right)^\gamma \frac{L(I_j)}{L(n)} X_{I_j}^{(r),j} + \sum_{j=a+2}^{K} \left(\frac{I_j}{n}\right)^\gamma \frac{L(I_j)}{L(n)} X_{I_j}^{(b),j} + T_r^{(n)}$$

$X_n^{(b)} \stackrel{d}{=}$ similarly

Fixed-point equations

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Limit

$$X^{(r)} \stackrel{d}{=} \sum_{j=1}^{a+1} D_j^{\gamma} X^{(r),j} + \sum_{j=a+2}^{K} D_j^{\gamma} X^{(b),j} + T_r$$

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Calls for recursive methods: **Contraction method**

More colors

Types (colors) $1, \ldots, m$.

$R_n^{[j]}$: # type 1 balls, n steps, start with one type j ball.

$$\mathbb{E}\,[R_n^{[j]}] = \begin{cases} c_\mu n + d_j n^\lambda + o(n^\lambda) & \lambda > 1/2 \\ c_\mu n + o(\sqrt{n}) & \\ c_\mu n + \Re(\kappa_j n^{i\mu}) n^\lambda + o(n^\lambda) & \lambda > 1/2 \end{cases}$$

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System of limits:

$$X^{[j]} \stackrel{d}{=} \sum_{i=1}^{m} \sum_{r \in J_{ij}} (D_r)^\gamma X^{[i],r} + b^{[j]}, \quad j = 1, \ldots, m.$$

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Limit map

$$X^{(r)} \stackrel{d}{=} \sum_{j=1}^{a+1} D_j^{\gamma} X^{(r),j} + \sum_{j=a+2}^{K} D_j^{\gamma} X^{(b),j} + T_r$$

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Corresponding limit map:

$$\begin{aligned} T : \mathcal{M} \times \mathcal{M} &\to \mathcal{M} \times \mathcal{M} \\ (\mu, \nu) &\mapsto \Big(\mathcal{L}\Big(\sum_{j=1}^{a+1} D_j^{\gamma} W_j + \sum_{j=a+2}^{K} D_j^{\gamma} Z_j + T_r \Big), \\ &\qquad \mathcal{L}\Big(\sum_{j=1}^{c} D_j^{\gamma} W_j + \sum_{j=c+1}^{K} D_j^{\gamma} Z_j + T_b \Big) \Big), \end{aligned}$$

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where the W_j, Z_j are independent and $\mathcal{L}(W_j) = \mu$, $\mathcal{L}(Z_j) = \nu$ for all j.

Metrics

Useful metrics on $\mathcal{M}$:

ℓ_p: minimal L_p-metric

ζ_s: Zolotarev metric

On appropriate subspaces of $\mathcal{M} \times \mathcal{M}$:

$$\ell_p^\vee((\mu_1, \nu_1), (\mu_2, \nu_2)) := \max\{\ell_p(\mu_1, \mu_2), \ell_p(\nu_1, \nu_2)\}$$

$$\zeta_s^\vee((\mu_1, \nu_1), (\mu_2, \nu_2)) := \max\{\zeta_s(\mu_1, \mu_2), \zeta_s(\nu_1, \nu_2)\}$$

Metrics

Useful metrics on $\mathcal{M}$:

ℓ_p: minimal L_p-metric

ζ_s: Zolotarev metric

On appropriate subspaces of $\mathcal{M} \times \mathcal{M}$:

$$\ell_p^\vee((\mu_1,\nu_1),(\mu_2,\nu_2)) := \max\{\ell_p(\mu_1,\mu_2),\ell_p(\nu_1,\nu_2)\}$$

$$\zeta_s^\vee((\mu_1,\nu_1),(\mu_2,\nu_2)) := \max\{\zeta_s(\mu_1,\mu_2),\zeta_s(\nu_1,\nu_2)\}$$

$$\mathbb{E}\left[R_n^{[j]}\right] = \begin{cases} c_\mu n + d_j n^\lambda + o(n^\lambda) & \lambda > 1/2 \qquad \gamma = \lambda \\ c_\mu n + o(\sqrt{n}) & \qquad \gamma = 1/2 \\ c_\mu n + \Re(\kappa_j n^{i\mu}) n^\lambda + o(n^\lambda) & \lambda > 1/2 \qquad \gamma = \lambda + i\mu \end{cases}$$

Another example

Random replacement:

$$\begin{bmatrix} B_\alpha & 1 - B_\alpha \\ 1 - B_\beta & B_\beta \end{bmatrix}, \qquad \alpha, \beta \in [0, 1]$$

B_α: Bernoulli(α)

B_β: Bernoulli(β)

Smythe & Rosenberger (1995), Smythe (1996), Bai et al. (1999, 2002), Janson (2004).

Recurrences

$R_n^{(r)}$: # red balls, initial red, $R_n^{(b)}$: # red balls, initial blue.

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$$R_n^{(r)} \stackrel{d}{=} R_{I_n}^{(r)} + B_\alpha \widehat{R}_{n+1-I_n}^{(r)} + (1 - B_\alpha)\widehat{R}_{n+1-I_n}^{(b)}$$

$$R_n^{(b)} \stackrel{d}{=} R_{I_n}^{(b)} + B_\beta \widehat{R}_{n+1-I_n}^{(b)} + (1 - B_\beta)\widehat{R}_{n+1-I_n}^{(r)}$$

$\mathbf{I_n}$ uniform$\{1, \ldots, \mathbf{n}\}$ distributed.

Limit equations

Case $\alpha + \beta \le 3/2$:

$$X^{(r)} \overset{d}{=} \sqrt{U} X^{(r)} + \sqrt{1-U} B_\alpha \widehat{X}^{(r)} + \sqrt{1-U}(1 - B_\alpha) X^{(b)},$$

$$X^{(b)} \overset{d}{=} \sqrt{U} X^{(b)} + \sqrt{1-U} B_\beta \widehat{X}^{(b)} + \sqrt{1-U}(1 - B_\alpha) X^{(r)}.$$

Case $\alpha + \beta > 3/2$: Set $\gamma := \alpha + \beta - 1$.

$$X^{(r)} \overset{d}{=} U^\gamma X^{(r)} + (1-U)^\gamma B_\alpha \widehat{X}^{(r)} + (1-U)^\gamma (1 - B_\alpha) X^{(b)} + T_r,$$

$$X^{(b)} \overset{d}{=} U^\gamma X^{(b)} + (1-U)^\gamma B_\beta \widehat{X}^{(b)} + (1-U)^\gamma (1 - B_\alpha) X^{(r)} + T_b.$$

In both systems $X^{(r)}$, $\widehat{X}^{(r)}$, $X^{(b)}$, $\widehat{X}^{(b)}$, U independent.

Bivariate formulation

$$R_n := \begin{pmatrix} R_n^{(r)} \\ R_n^{(b)} \end{pmatrix}.$$

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$$R_n \stackrel{d}{=} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} R_{I_n} + \begin{bmatrix} B_\alpha & 1 - B_\alpha \\ 1 - B_\beta & B_\beta \end{bmatrix} \widehat{R}_{n+1-I_n}$$

$(R_j)_j, (\widehat{R}_j)_j, B_\alpha, B_\beta, I_n$ independent.

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$(R_j)_j, (\widehat{R}_j)_j, B_\alpha, B_\beta, I_n$ independent.

Coupling:

Urns starting with red resp. blue ball are coupled.

Limit equation

(I) $\alpha + \beta \leq 3/2$.

$$X_n := \frac{1}{\sqrt{n}}(R_n - \mathbb{E}\, R_n)$$

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Bivariate normal distribution solves.

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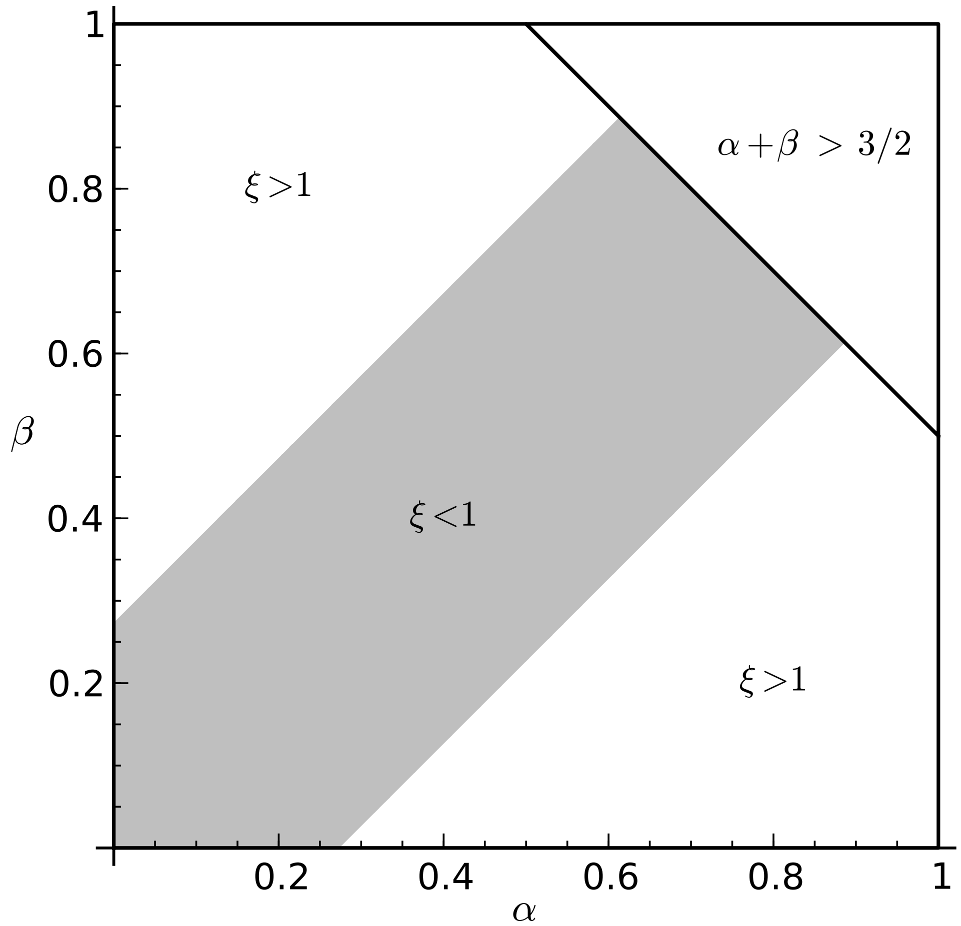
Bivariate normal distribution solves.

Do not have contraction in the whole range!

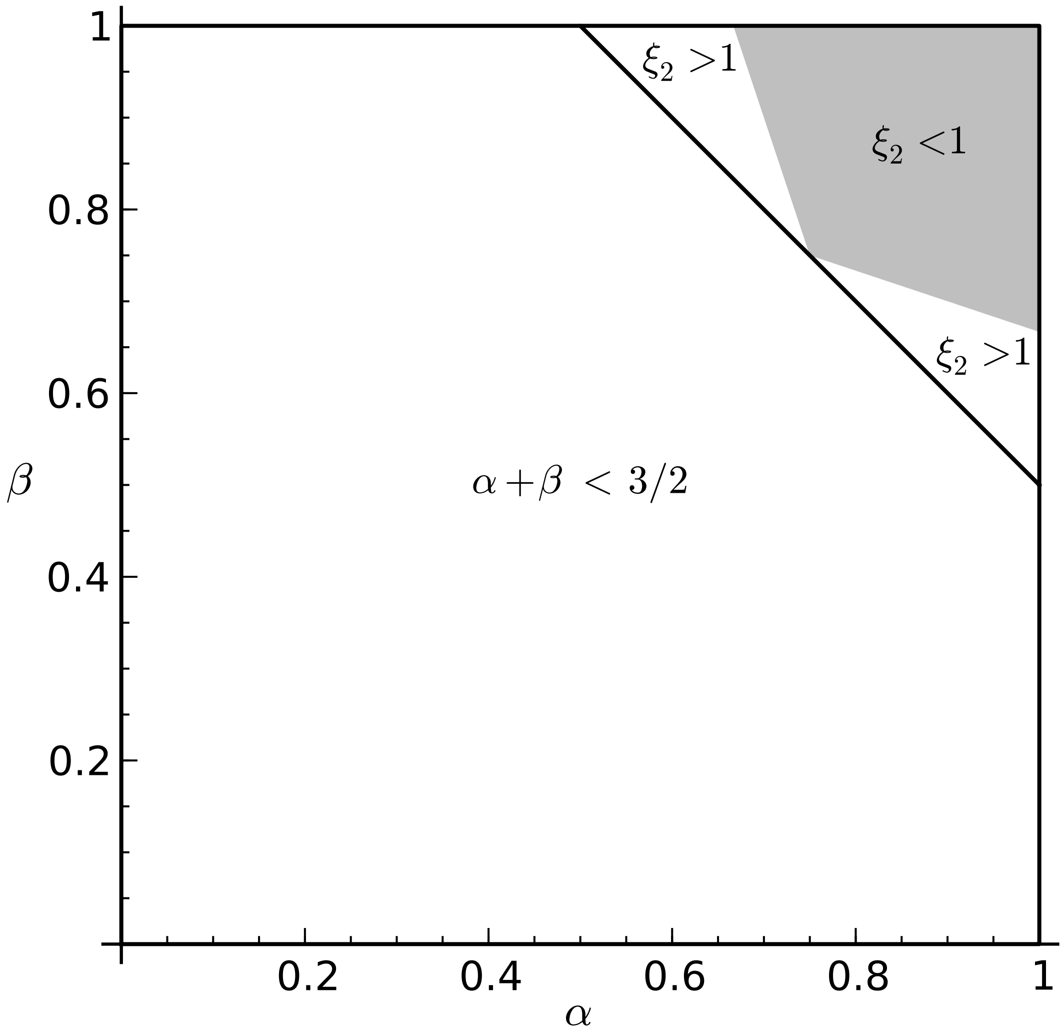
Contraction condition

$$\mathbb{E}\left[U^{3/2}\right] + \mathbb{E}\left[(1-U)^{3/2}\right]\mathbb{E}\left[\left\| \begin{matrix} B_\alpha & 1-B_\alpha \\ 1-B_\beta & B_\beta \end{matrix} \right\|_{\text{op}}^3\right] < 1.$$



Contraction condition

Non-normal case $\alpha + \beta > 3/2$:



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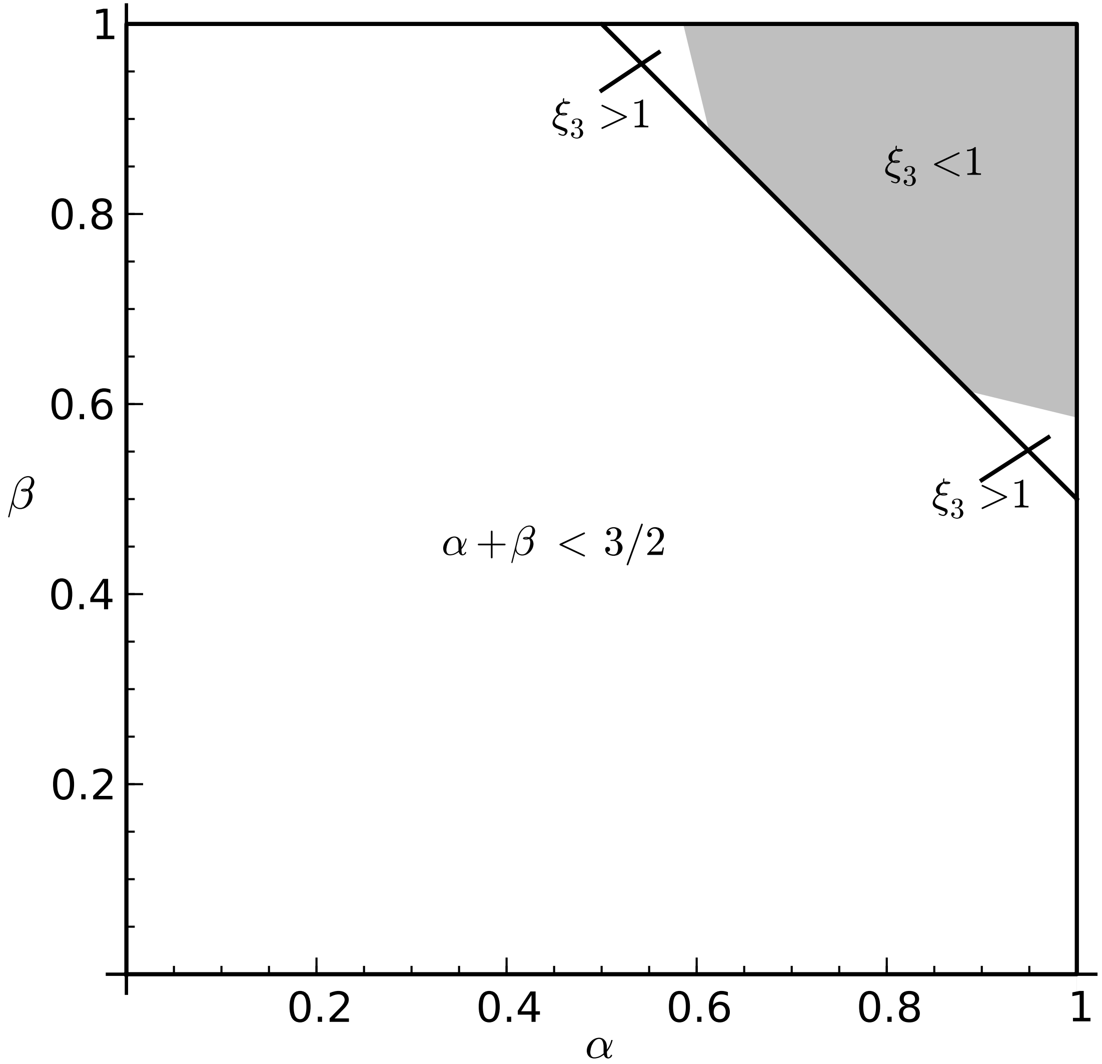

Systems versus multivariate

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System formulation:

$$X^{(r)} \stackrel{d}{=} \sqrt{U} X^{(r)} + \sqrt{1-U} B_\alpha \widehat{X}^{(r)} + \sqrt{1-U}(1-B_\alpha) X^{(b)},$$

$$X^{(b)} \stackrel{d}{=} \sqrt{U} X^{(b)} + \sqrt{1-U} B_\beta \widehat{X}^{(b)} + \sqrt{1-U}(1-B_\alpha) X^{(r)}.$$

Work space: $\mathcal{M}(\mathbb{R}) \times \mathcal{M}(\mathbb{R})$ (Precise: $\mathcal{M}_3^{\mathbb{R}}(0,1) \times \mathcal{M}_3^{\mathbb{R}}(0,1)$)

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Bivariate formulation:

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \overset{d}{=} \sqrt{U}\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} + \sqrt{1-U}\begin{bmatrix} B_\alpha & 1-B_\alpha \\ 1-B_\beta & B_\beta \end{bmatrix}\begin{pmatrix} \widehat{X}_1 \\ \widehat{X}_2 \end{pmatrix}$$

Work space: $\mathcal{M}(\mathbb{R} \times \mathbb{R})$ (Precise: $\mathcal{M}_3^{\mathbb{R}^2}(0,\mathrm{Id}_2)$).

Cyclic urns

$m \geq 2$ types (colors)

$$R = \begin{bmatrix} 0 & 1 & & & 0 \\ & 0 & 1 & & \\ & & 0 & & \\ & & & \ddots & 1 \\ 1 & & & & 0 \end{bmatrix}$$

$R_n^{[j]}$: # type 1 balls after n steps starting with one type j ball.

Janson (1983, 2004, 2006), Pouyanne (2005, 2008)

Recurrences

$R_n^{[j]}$: # type 1 balls after n steps starting with one type j ball.

$$\begin{aligned} R_n^{[1]} &\stackrel{d}{=} R_{I_n}^{[1]} + R_{J_n}^{[2]}, \\ R_n^{[2]} &\stackrel{d}{=} R_{I_n}^{[2]} + R_{J_n}^{[3]}, \\ &\vdots \\ R_n^{[m]} &\stackrel{d}{=} R_{I_n}^{[m]} + R_{J_n}^{[1]}, \end{aligned}$$

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I_n: uniform on $\{0, \ldots, n-1\}$.

$$\mathbb{E}\left[R_n^{[j]}\right] = \begin{cases} \frac{n}{m} + o(\sqrt{n}), & 2 \le m \le 5 \\ \frac{n}{m} + \Theta(\sqrt{n}), & m = 6 \\ \frac{n}{m} + \Re(\kappa_j n^{i\mu}) n^{\lambda} + o(n^{\lambda}), & m \ge 7. \end{cases}$$

Limit equations

$2 \le m \le 6$: With U unif$[0,1]$:

$$\begin{aligned} X^{[1]} &\stackrel{d}{=} \sqrt{U}X^{[1]} + \sqrt{1-U}X^{[2]}, \\ X^{[2]} &\stackrel{d}{=} \sqrt{U}X^{[2]} + \sqrt{1-U}X^{[3]}, \\ &\vdots \\ X^{[m]} &\stackrel{d}{=} \sqrt{U}X^{[m]} + \sqrt{1-U}X^{[1]}, \end{aligned}$$

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$m \geq 7$: With $\omega := e^{2\pi i/m}$:

$$\begin{aligned} X^{[1]} &\stackrel{d}{=} U^{\omega}X^{[1]} + (1-U)^{\omega}X^{[2]}, \\ X^{[2]} &\stackrel{d}{=} U^{\omega}X^{[2]} + (1-U)^{\omega}X^{[3]}, \\ &\vdots \\ X^{[m]} &\stackrel{d}{=} U^{\omega}X^{[m]} + (1-U)^{\omega}X^{[1]}. \end{aligned}$$

Periodic case $m \geq 7$

System reduces:

$$X \stackrel{d}{=} U^\omega X + \omega(1-U)^\omega X' \quad \text{in} \quad \mathcal{M}_2^{\mathbb{C}}\left(\frac{2}{m\Gamma(\omega+1)}\right),$$

(cf. Janson ALEA06)

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Then

$$X^{[j]} \stackrel{d}{=} \omega^{j-1} X, \quad j = 1, \ldots, m.$$

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(cf. Janson ALEA06)

Then

$$X^{[j]} \stackrel{d}{=} \omega^{j-1} X, \qquad j = 1, \ldots, m.$$

Asymptotic periodic behavior:

$$\ell_2\left(\frac{R_n^{[j]} - \frac{n}{m}}{n^{\lambda}}, \Re\left(e^{i\left(\mu \ln(n) + 2\pi \frac{j-1}{m}\right)} X\right)\right) \to 0 \quad (n \to \infty).$$