Phase Transition of Inhomogeneous Random Graphs

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Joint work with Vlady Ravelomanana

Introduction

- The inhomogeneous graphs model encodes several tractable SAT and CSP problems
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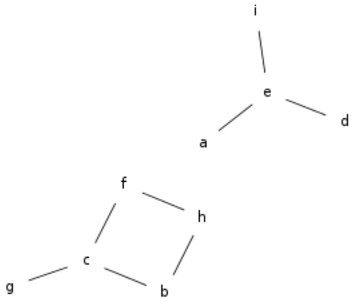
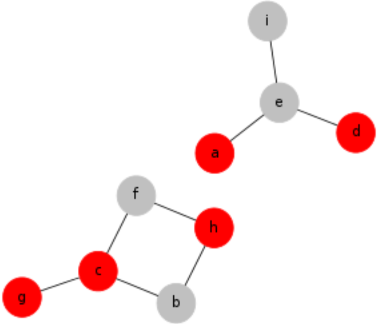
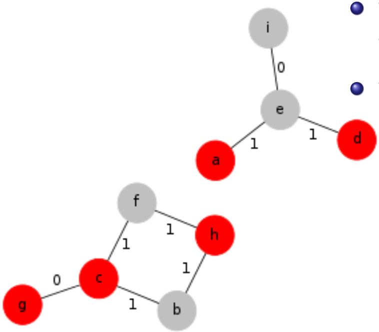
- Phase transition of the modeled problems
  probability for a graph to be bipartite [Pittel Yeum 10],
  probability of satisfiability of a quantified 2-XOR-SAT formula *[Creignou Daudé Egly 07]*

Bipartite Graphs *[Pittel Yeum 10]*



- Each vertex v receives a color $c(v)$,





- Each vertex v receives a color $c(v)$,
- the edges are weighted according to the color of their ends using $R = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$,
- weight $\frac{1}{2}$ for each connected component.

$$\text{weight}(c(G)) := \left(\frac{1}{2}\right)^{\text{cc}(G)} \prod_{(a,b) \in E(G)} R_{c(a),c(b)}$$

weight $= 0$

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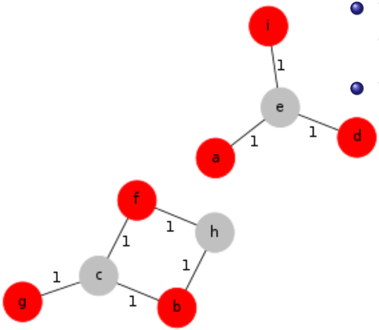

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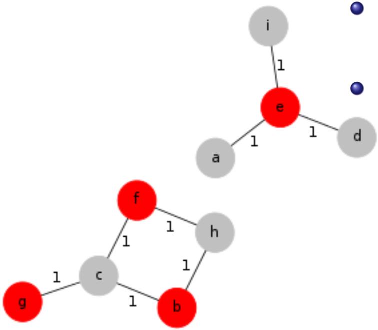

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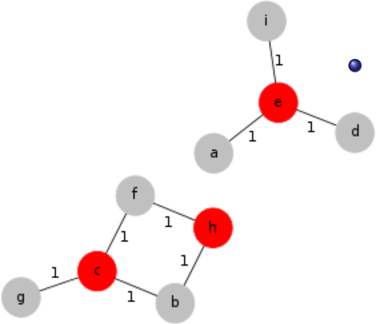

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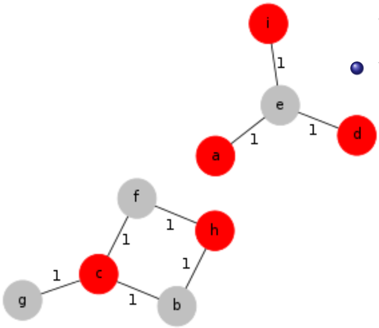

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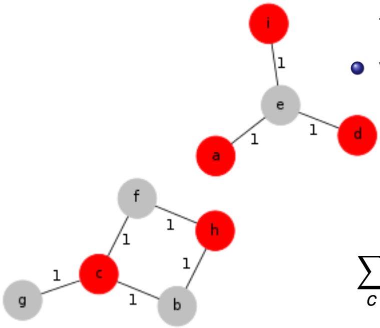

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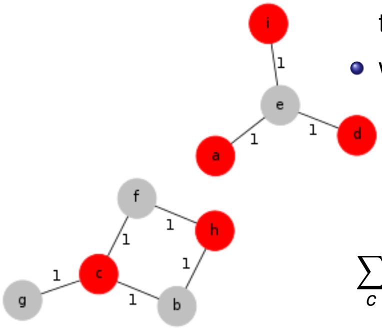

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The number of (n,m)-bipartite graphs is

$$g_{\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), \frac{1}{2}}(n,m) := \sum_{(n,m)\text{-graph } G} \sum_{c} \text{weight}(c(G)).$$

Inhomogeneous Graphs *[Söderberg 02] [Bollobàs Janson Riordan 07]*

$R \in \mathrm{Sym}_{q\times q}(\mathbb{R}_{\geq 0})$ and $\sigma > 0$. A (R,σ)-graph is:

- a vertex colored graph $c(G)$,
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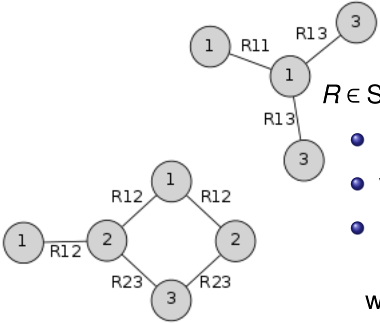

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$$\mathrm{weight} = \sigma^2 R_{1,1} R_{1,2}^3 R_{1,3}^2 R_{2,3}^2$$

Quantified 2-Xor-Sat Formulas *[Creignou Daudé Egly 07]*

$$\forall x, y,\ \exists a, b, \dots, h,\ a \oplus b = x,\ a \oplus h = y,\ a \oplus c = x,$$
$$b \oplus e = x,\ d \oplus f = x,\ d \oplus g = y,\ e \oplus h = y$$

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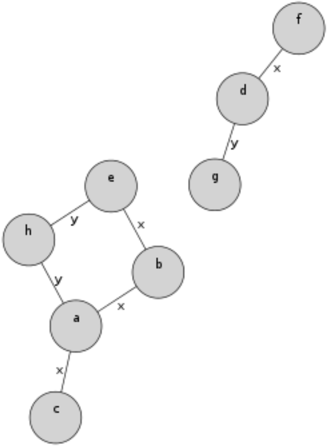

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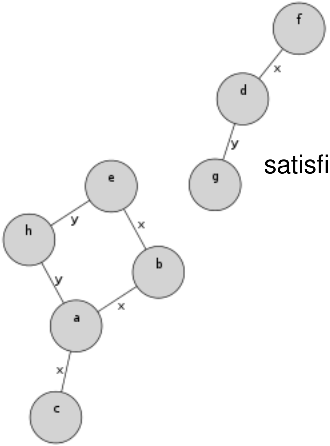

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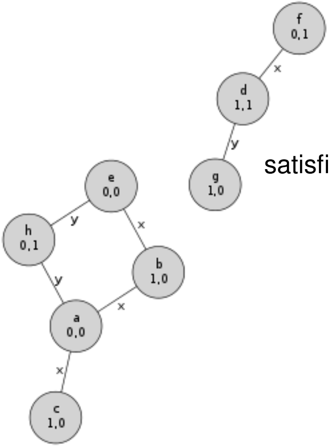
satisfiable iff each cycle contains an even number of x and y.



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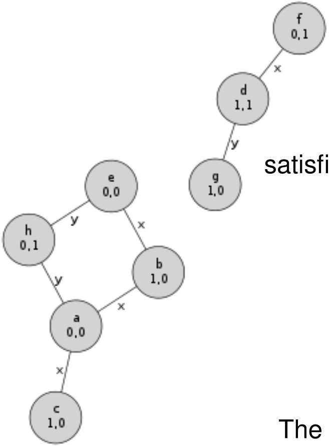

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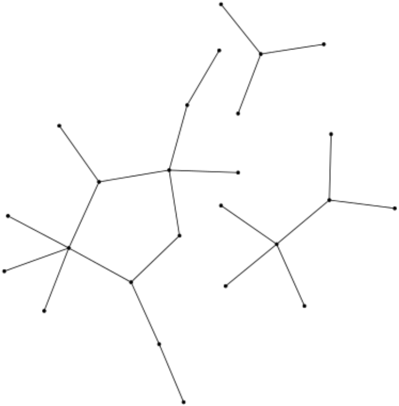
The number of satisfiable quantified 2-Xor-Sat formulas with n existantial variables and m clauses is $g_{R,\sigma}(n, m)$.

Sub-Critical Density of Edges

When $\frac{m}{n} < c(1-\epsilon)$ and $n \to \infty$, with high probability a (n,m)-(R,σ)-graph consists of trees and unicycle components.



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rooted tree $$T_i(z) = z\exp(\overleftarrow{R_i}\,\overrightarrow{T}(z))$$

Symbolic method

$$z\partial\overrightarrow{T} = (I - \mathrm{diag}(\overrightarrow{T})R)^{-1}\,\overrightarrow{T}$$

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Drmota-Lalley-Wood Theorem

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unrooted tree $U = \overleftrightarrow{1}\,\overrightarrow{T} - \frac{1}{2}\overleftarrow{T} R \overrightarrow{T}$ $\sim u_0 + u_2(1-\frac{z}{\rho}) + u_3(1-\frac{z}{\rho})^{3/2}$

Dissymmetry Theorem

$$z\partial U = T_1 + \ldots + T_q$$

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linear algebra

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$$g_{R,\sigma}(n,m) \sim n![z^n]\frac{(\sigma U)^{n-m}}{(n-m)!}e^{\sigma V}$$

Large Power scheme *[Flajolet Sedgewick 09]*: one dominant saddle point.

Critical Case

When $\frac{m}{n} = c(1 + \mu n^{-1/3})$ where $\mu = O(1)$, with high probability a (n, m)-(R, σ)-graph consists of

- trees and unicycle components,
- a cubic multigraph where the vertices are replaced by rooted trees and the edges by paths of trees.

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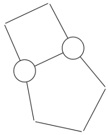

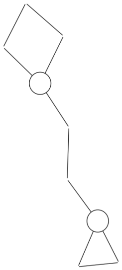

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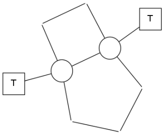

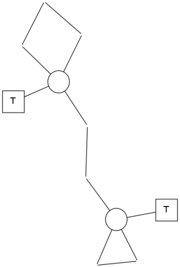

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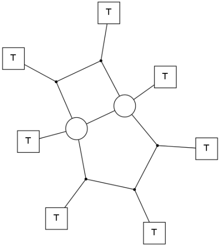

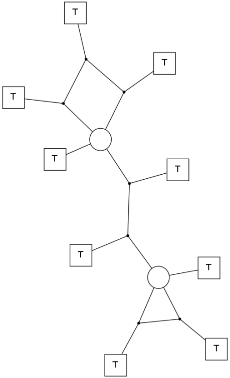

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In a cubic graph, each vertex owns $3/2$ edges.
For the ordinary graphs, the gf of the developped cubic part is

$$\mathrm{GF}_{\text{cubic}}\left(z \leftarrow T(z)\left(\frac{1}{1 - T(z)}\right)^{3/2}\right)$$

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$$g_{R,\sigma}(n,m) \sim n![z^n] \sum_k \frac{(\sigma U)^{k+n-m}}{(k+n-m)!} \exp(\sigma V) \frac{e_k^{(\sigma)}(T_1 p_1^3 + \cdots T_q p_q^3)^{2k}}{\det(I - \operatorname{diag}(\vec{T})R)^{3k}}$$

Coallescence of two saddle points at the dominant singularity of $\vec{T}(z)$ *[Janson Knuth Łuczak Pittel 93]* or *[Banderier Flajolet Schaeffer Soria 01]*.

Results

New proof of the result of Pittel and Yeum on the probability of bipartiteness, new result on the probability of satisfiability of quantified 2-Xor-Sat formulas:

$$\frac{2m}{n} < 1-\epsilon \ : \mathbb{P}(\text{sat}) \sim \frac{\left(1-\frac{2m}{n}\right)^{3/8}}{\left(1+\frac{2m}{n}\right)^{1/8}} \sqrt{1-\frac{m}{n}},$$

$$\frac{2m}{n} = 1 + \mu n^{-1/3} \ : \mathbb{P}(\text{sat}) \sim \frac{\Phi_{1/4}(\mu)}{(2n)^{1/8}}.$$

$$\Phi_\sigma(\mu) = \sqrt{2\pi} \sum_k \frac{e_k^{(\sigma)}}{4^k} A(3k+\sigma/2, \mu)$$

$$e_k^{(\sigma)} = [z^{2k}] \left(\sum_n \frac{(6n)! z^{2n}}{(2n)!(3n)! 2^n (3!)^n} \right)^\sigma$$

$$A(y,\mu) = \frac{e^{-\mu^3/6}}{3^{(y+1)/3}} \sum_k \frac{(3^{2/3}\mu/2)^k}{k!\,\Gamma((y+1-2k)/3)}$$

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