Average Redundancy of the Shannon Code for Markov Sources

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Dedicated to PHILIPPE FLAJOLET

Outline

1. Source Coding
2. Redundancy: Known Sources
3. Shannon and Huffman Redundancy for Memoryless Sources
4. Shannon Coding Redundancy for Markov Sources

Source Coding

A source code is a bijective mapping

$$C : \mathcal{A}^* \to \{0,1\}^*$$

from sequences over the alphabet $\mathcal{A}$ to set $\{0,1\}^*$ of binary sequences.

The basic problem of source coding (i.e., *data compression*) is to find codes with shortest descriptions (lengths) either on *average* or for *individual sequences*.

Three Basic Types of Source Coding:

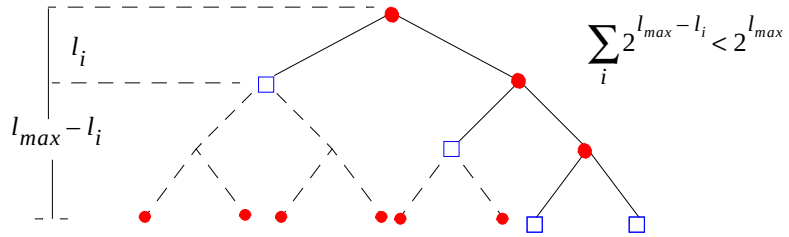
- Fixed-to-Variable (**FV**) length codes (e.g., Huffman and Shannon codes).
- Variable-to-Fixed (**VF**) length codes (e.g., Tunstall and Khodak codes).
- Variable-to-Variable (**VV**) length codes (e.g., Khodak VV code).

Prefix Codes

Prefix code is such that no codeword is a prefix of another codeword.
We write:
$P(x)$ be the probability of $x \in \mathcal{A}^*$;
$L(C, x)$ be the code length for the source sequence $x \in \mathcal{A}^*$;
$H(P) = -\sum_{x \in \mathcal{A}^*} P(x) \lg P(x)$ for entropy.

Kraft's Inequality

A binary code is a prefix code iff the code lengths $\ell_1, \ell_2, \ldots, \ell_N$ satisfy



$$\sum_{i=1}^{N} 2^{-\ell_i} \leq 1.$$

Shannon First Theorem

For any prefix code the average code length $\mathbf{E}[L(C, X)]$ cannot be smaller than the entropy of the source $H(P)$, that is,

$$\mathbf{E}[L(C_n, X)] \geq H(P).$$

Exercise: There exists at least one sequence $\tilde{x}_1^n$ such that $L(\tilde{x}_1^n) \geq -\log_2 P(\tilde{x}_1^n)$.

Redundancy

Known Source P.

The pointwise redundancy $R(x)$ and the average redundancy $\bar{R}$:

$$\begin{aligned} R(x) &= L(C,x) + \lg P(x) \\ \bar{R} &= \mathbf{E}[L(C,X)] - H(P) \geq 0 \end{aligned}$$

Optimal Code:

$$\min_L \sum_x L(x)P(x) \quad \text{subject to} \quad \sum_x 2^{-L(x)} \leq 1.$$

Solution: By Lagrangian multipliers we find $L^{opt}(x) = -\lg P(x)$.

The smaller the redundancy is, the better (closer to the optimal) the code is.

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Redundancy for Huffman's Code

We consider **fixed-to-variable length codes**: Shannon & Huffman codes.

For a known source P, we consider fixed length sequences $x_1^n = x_1 \dots x_n$.

Huffman Code: The following optimization problem

$$\bar{R}_n = \min_{C_n \in \mathcal{C}} \mathbf{E}_{x_1^n}[L(C_n, x_1^n) + \log_2 P(x_1^n)].$$

is solved by Huffman's code.

We review first the average redundancy for a binary memoryless sources with p denoting the probability of generating "0" and $q = 1 - p$.

In 1994 Stubley proposed the following for Huffman's average redundancy

$$\bar{R}_n^H = 2 - \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} \langle \alpha k + \beta n \rangle - 2 \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} 2^{-\langle \alpha k + \beta n \rangle} + o(1).$$

where

$$\alpha = \log_2 \left(\frac{1-p}{p} \right), \quad \beta = \log_2 \left(\frac{1}{1-p} \right)$$

and $\langle x \rangle = x - \lfloor x \rfloor$ is the fractional part of x.

Main Result

Theorem 1 (W.S., 2000). *Consider the Huffman block code of length n over a binary memoryless source with $p < \frac{1}{2}$. Then as $n \to \infty$*

$$\bar{R}_n^H = \begin{cases} \frac{3}{2} - \frac{1}{\ln 2} + o(1) \approx 0.057304 & \alpha \quad \text{irrational} \\ \frac{3}{2} - \frac{1}{M}\left(\langle \beta M n \rangle - \frac{1}{2}\right) - \frac{1}{M(1-2^{-1/M})} 2^{-\langle n\beta M \rangle / M} + O(\rho^n) & \alpha = \frac{N}{M} \end{cases}$$

where N, M are integers such that $\gcd(N, M) = 1$ and $\rho < 1$.

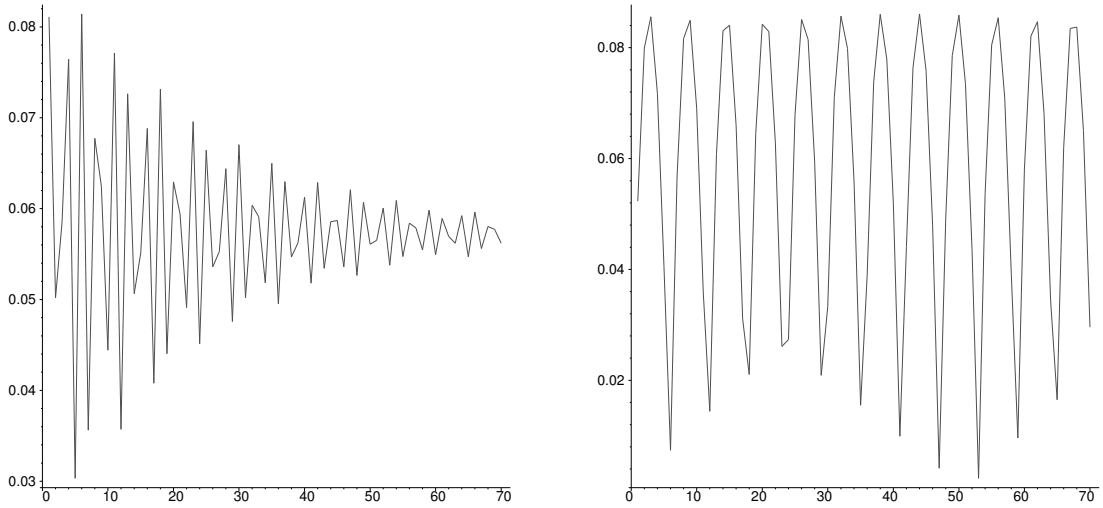

Figure 1: The average redundancy of Huffman codes versus block size n for: (a) irrational $\alpha = \log_2(1-p)/p$ with $p = 1/\pi$; (b) rational $\alpha = \log_2(1-p)/p$ with $p = 1/9$.

Why Two Modes: Shannon Code

Consider the Shannon code that assigns the length

$$L(C_n^S, x_1^n) = \lceil -\lg P(x_1^n) \rceil$$

to the source sequence x_1^n. Observe that

$$P(x_1^n) = p^k (1-p)^{n-k}$$

where p is **known** probability of generating 0 and k is the number of 0s.

The Shannon code redundancy is

$$\begin{aligned} \bar{R}_n^S &= \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} \left(\lceil -\log_2(p^k (1-p)^{n-k}) \rceil + \log_2(p^k (1-p)^{n-k}) \right) \\ &= 1 - \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} \langle \alpha k + \beta n \rangle \end{aligned}$$

where $\langle x \rangle = x - \lfloor x \rfloor$ is the fractional part of x, and

$$\alpha = \log_2 \left(\frac{1-p}{p} \right), \quad \beta = \log_2 \left(\frac{1}{1-p} \right).$$

Sketch of Proof

We need to understand asymptotic behavior of the following sum (cf. Bernoulli distributed sequences modulo 1)

$$\sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} f(\langle \alpha k + y \rangle)$$

for fixed p and some Riemann integrable function $f : [0,1] \to \mathbf{R}$.

Lemma 1. *Let $0 < p < 1$ be a fixed real number and α be an irrational number. Then for every Riemann integrable function $f : [0,1] \to \mathbb{R}$*

$$\lim_{n\to\infty} \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} f(\langle \alpha k + y \rangle) = \int_0^1 f(t)\, dt,$$

where the convergence is uniform for all shifts $y \in \mathbb{R}$.

Lemma 2. *Let $\alpha = \frac{N}{M}$ be a rational number with $\gcd(N, M) = 1$. Then for bounded function $f : [0,1] \to \mathbb{R}$*

$$\sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} f(\langle \alpha k + y \rangle) = \frac{1}{M} \sum_{l=0}^{M-1} f\left(\frac{l}{M} + \frac{\langle My \rangle}{M}\right) + O(\rho^n)$$

uniformly for all $y \in \mathbb{R}$ and some $\rho < 1$.

Shannon Redundancy – Rational Case

Assume $\alpha = N/M$ where $\gcd(N, M) = 1$. Denote $p_{n,k} = \binom{n}{k} p^k q^{n-k}$.

$$S_n = \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} \left\langle k\frac{N}{M} + \beta n \right\rangle = \sum_{\ell=0}^{M-1} \sum_{m:\ k=\ell+mM\le n} p_{n,k} \left\langle \ell\frac{N}{M} + N + \beta n \right\rangle$$

$$= \sum_{\ell=0}^{M-1} \left\langle \frac{\ell}{M} + \beta n \right\rangle \sum_{m:\ k=\ell+mM\le n} p_{n,k}.$$

Lemma 3. *For fixed $\ell \le M$ and M, there exist $\rho < 1$ such that*

$$\sum_{m:\ k=\ell+mM\le n} \binom{n}{k} p^k (1-p)^{n-k} = \frac{1}{M} + O(\rho^n).$$

Proof. Let $\omega_k = e^{2\pi i k/M}$ for $k = 0, 1, \ldots, M-1$ be the Mth root of unity.

$$\frac{1}{M} \sum_{k=0}^{M-1} \omega_k^n = \begin{cases} 1 & \text{if } M|n \\ 0 & \text{otherwise.} \end{cases}$$

where $M|n$ means that M divides n. Then

$$\sum_{m:\ k=\ell+mM\le n} \binom{n}{k} p^k q^{n-k} = \frac{1 + (p\omega_1 + q)^{n-\ell} + .. + (p\omega_{M-1} + q)^{n-\ell}}{M} = \frac{1}{M} + O(\rho^n),$$

since $|(p\omega_r + q)| = p^2 + q^2 + 2pq\cos(2\pi r/M) < 1$ for $r \neq 0$.

Finishing the Rational Case

We shall use the following Fourier series; for real x

$$\langle x \rangle = \frac{1}{2} - \sum_{m=1}^{\infty} \frac{\sin 2\pi m x}{m\pi} = \frac{1}{2} - \sum_{m \in \mathbf{Z} - \{0\}} c_m e^{2\pi i m x}, \quad c_m = -\frac{i}{2\pi m},$$

Continuing the derivation and using the above lemma we obtain

$$\begin{aligned} S_n &= \frac{1}{M} \sum_{\ell=0}^{M-1} \left(\frac{1}{2} - \sum_{m \neq 0} c_m e^{2\pi i m(\ell/M + \beta n)} \right) = \frac{1}{2} - \sum_{m \neq 0} c_m e^{2\pi i m n \beta} \frac{1}{M} \sum_{\ell=0}^{M-1} e^{2\pi i m \frac{\ell}{M}} \\ &= \frac{1}{2} - \frac{1}{M} \sum_{m = kM \neq 0} c_{kM} e^{2\pi i k M \beta n} = \frac{1}{2} - \frac{1}{M} \left(\frac{1}{2} - \langle \beta n M \rangle \right). \end{aligned}$$

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Markov Sources

Source sequence $X_1, X_2, \ldots,$ over alphabet $\mathcal{A} = \{1, 2, \ldots, r\}$ is generated by a first–order Markov chain with a given matrix

$$P = \{p(j|k)\}_{j,k=1}^{r}.$$

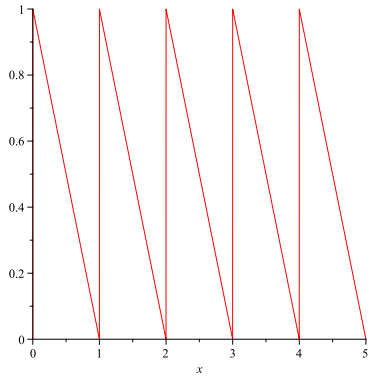
with initial state probabilities p_k, $k = 1, 2, \ldots, r$;
stationary state probabilities π_k, $k = 1, 2, \ldots, r$.

For $x^n = (x_1, \ldots, x_n) \in \mathcal{A}^n$ under the given Markov source, is

$$P(x^n) = p_{x_1} \prod_{t=2}^{n} p(x_t|x_{t-1}).$$

The average redundancy of the Shannon code is defined as

$$R_n = \boldsymbol{E}[\lceil -\log P(X^n) \rceil + \log P(X^n)] = \mathbf{E}[\varrho(-\log P(X^n))].$$



$$\varrho(u) = \lceil u \rceil - u.$$

Main Result for Markov Sources

Theorem 2 (Merhav & W.S.). *Consider the Shannon code of length n for an aperiodic and irreducible Markov source. Define*

$$\alpha_{jk} = \log\left[\frac{p(j|1)p(j|j)}{p(k|1)p(j|k)}\right], \quad j,k \in \{1,2,\ldots,r\}.$$

(a) If not all $\{\alpha_{jk}\}$ are rational, then

$$R_n = \frac{1}{2} + o(1).$$

(b) If all $\{\alpha_{jk}\}$ are rational, then let

$$\zeta_{jk}(n) = M[-(n-1)\log p(1|1) + \log p(j|1) - \log p(k|1) - \log p_j],$$

and

$$\Omega_n = \frac{1}{2}\left(1 - \frac{1}{M}\right) + \frac{1}{M}\sum_{j=1}^{r}\sum_{k=1}^{r} p_j \pi_k \varrho[\zeta_{jk}(n)],$$

M is the smallest common multiple of the denominators of $\{\alpha_{jk}\}$. Then, there exists a positive sequence $\xi_n \to 0$

$$R_n \leq \Omega_n + \frac{1}{M}\sum_{j=1}^{r}\sum_{k=1}^{r} p_j \pi_k \mathcal{I}\{\varrho[\zeta_{jk}(n)] \notin (\xi_n, 1-\xi_n)\} + o(1),$$

$$R_n \geq \Omega_n - \frac{1}{M}\sum_{j=1}^{r}\sum_{k=1}^{r} p_j \pi_k \mathcal{I}\{\varrho[\zeta_{jk}(n)] \notin (\xi_n, 1-\xi_n)\} - o(1).$$

Sketch of Proof

1. We note that $\rho(u)$ has the following Fourier series expansion

$$\varrho(u) = \frac{1}{2} + \sum_{m \neq 0} a_m e^{2\pi i m u}, \qquad a_m = \frac{1}{2\pi m}$$

where $a_{m \cdot k} = a_m / k$ for integers m, k.

2. Since $R_n = \mathbf{E}[\varrho(\log P(X^n))]$ (for aperiodic irreducible MC) we have

$$R_n = \frac{1}{2} + \sum_{m \neq 0} a_m \boldsymbol{E}[e^{-2\pi i m \log P(X^n)}].$$

which we can re-write as

$$R_n = \frac{1}{2} + \sum_{m \neq 0} a_m \sum_{\mathbf{x} \in \mathcal{A}^n} \prod_{t=1}^{n} p(x_t | x_{t-1}) \exp\left[-2\pi i m \log p(x_t | x_{t-1})\right]$$

since $P(x^n) = p_{x_1} \prod_{t=2}^{n} p(x_t | x_{t-1})$.

Continuation

3. Define for $j, k = 1, \ldots, r$

$$a_{jk}(m) = p(k|j) \exp\left[-2\pi i m \log p(k|j)\right], \quad A_m = [a_{jk}(m)]_{j,k=1}^{r},$$

and r–dimensional column vectors:

$$\boldsymbol{c}_m = (p_1 \exp[-2\pi i m \log p_1)], \ldots, p_r \exp[-2\pi i m \log p_r])^T,$$

and $\mathbf{1} = (1, 1, \ldots, 1)^T$. Then

$$R_n = \frac{1}{2} + \sum_{m \neq 0} a_m \cdot \boldsymbol{c}_m^T A_m^{n-1} \mathbf{1}.$$

4. Let $\mathbf{l}_{i,m}$ and $\boldsymbol{r}_{i,m}$ be, respectively, the left eigenvector and the right eigenvector pertaining to the eigenvalue $\lambda_{i,m}$ of the matrix A_m such that $\rho(A_m) := |\lambda_{1,m}| \geq |\lambda_{2,m}| \geq \cdots |\lambda_{r,m}|$. By the spectral representation of matrices

$$A_m^{n-1} \mathbf{1} = \sum_{i=1}^{r} \lambda_{i,m}^{n-1} \cdot (\mathbf{l}_{i,m}^T, \mathbf{1}) \cdot \boldsymbol{r}_{i,m},$$

leading to

$$R_n = \frac{1}{2} + \sum_{m \neq 0} a_m \cdot \sum_{i=1}^{r} \lambda_{i,m}^{n-1} \cdot (\mathbf{l}_{i,m}^T, \mathbf{1}) \cdot (\boldsymbol{c}_m^T, \boldsymbol{r}_{i,m}).$$

Conclusions

5. If (a) all eigenvalues $\lambda_{i,m} < 1$, then by Fejer's theorem

$$R_n \to \frac{1}{2}.$$

(b) If some (largest) $\lambda_{i,m} = 1$, we have oscillatory provided

$$\rho(A_m) = \rho(P) = 1.$$

Lemma 4. *Let $F = \{f_{kj}\}$ and $G = \{g_{kj}\}$ be two $r \times r$ matrices. Assume that F is a real, non–negative and irreducible matrix, G is a complex matrix, and*

$$f_{kj} = |g_{kj}|, \quad k, j \in \{1, 2, \ldots, r\}.$$

Then

$$\rho(G) = \rho(F)$$

if and only if there exist real numbers s, and $w_1, \ldots, w_r$ such that

$$G = e^{2\pi i s} D F D^{-1},$$

where $D = diag\{e^{2\pi i w_1}, \ldots, e^{2\pi i w_r}\}$.

Thus $\rho(A_m) = \rho(P) = 1$ if and only if there exist s and $w_1, \ldots w_r$:

$$-m \log p(j|k) = (s + w_k - w_j) \bmod 1, \quad j, k = 1, \ldots, r,$$

where $x = y \bmod 1$ means $\langle x \rangle = \langle y \rangle$.

Extensions

We can extend our Theorem to irreducible and periodic Markov chains, but not further than this as example below shows.

Example 2. *Reducible Markov source*.
Consider the case $r = 2$ with the following transition matrix

$$P = \begin{pmatrix} 1-\alpha & \alpha \\ 0 & 1 \end{pmatrix}.$$

Assume also that $p_1 = 1$ and $p_2 = 0$. Direct computation shows that

$$R_n = \sum_{k=0}^{\infty} \alpha(1-\alpha)^k \varrho[-\log \alpha - k \log(1-\alpha)] + o(1).$$

We see then that there is *no oscillatory mode* in this case, as R_n always tends to a constant that depends on α.

That's It

THANK YOU