PROBABILISTIC ANALYSIS OF DIGITAL SEARCH TREES BUILT ON A GENERAL SOURCE

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This talk : Analysis of Digital Search Trees

when they are built on words emitted by a general source.

(I) Digital Search Trees and simple sources

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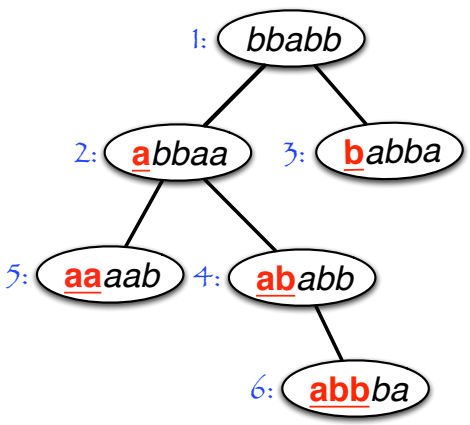

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$Y_1 = bbabb, Y_2 = abbaa,$
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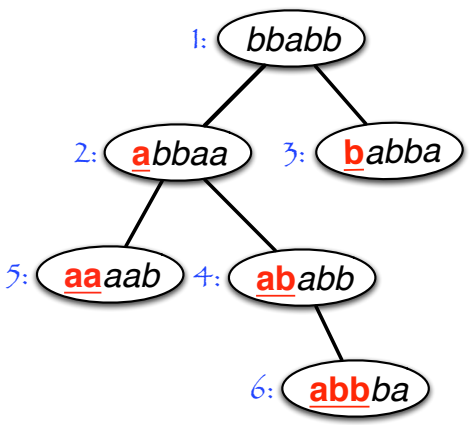

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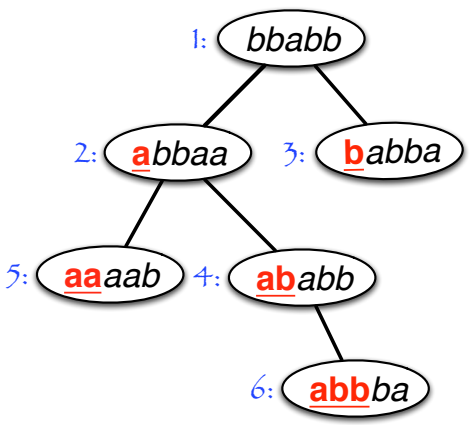

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Right [dst $(\mathcal{Y})$] := $\mathtt{dst}(\mathcal{Y}_{(b)})$.

Analysis of the `dst` structure $=$ Description of the shape of the `dst` built on a sequence of n words independently emitted from a source $\mathcal{S}$

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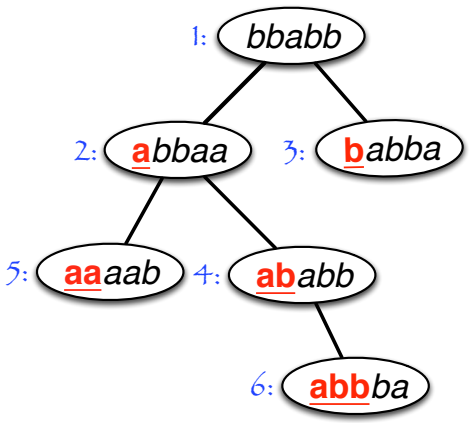

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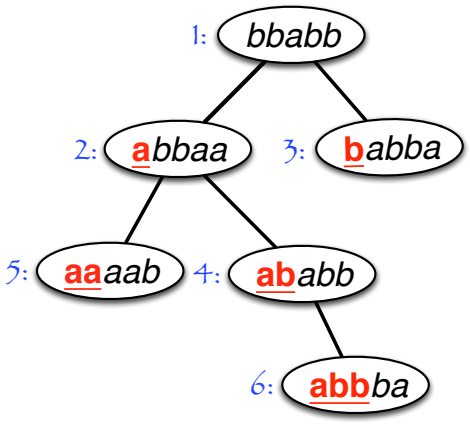

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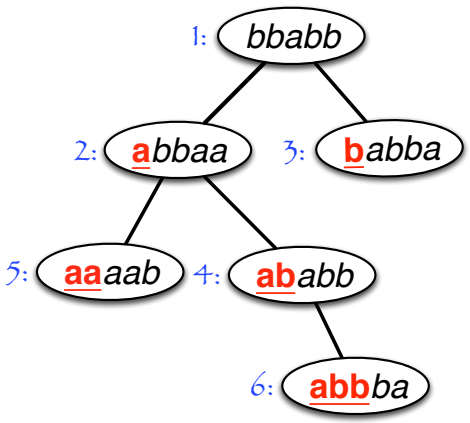

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Main object for the study: the Poisson gf $\quad B(z) := e^{-z}\sum_{n\geq 0} B_n(u)\frac{z^n}{n!}$

Simple sources

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A source:= a random mechanism which emits symbols from alphabet Σ,

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In both cases, $\lambda(s)$ is called the dominant eigenvalue of the source.
The position of the set $\mathcal{Z} := \{s, \lambda(s) = 1\}$ wrt the vertical line $\Re s = 1$
is essential. It is related to arithmetical properties of probabilities.

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Deal with it for extending the results for dst's to a general source

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For $u \in \Sigma^\star$ with $p_u \neq 0$, the source $\mathcal{S}_{(u)} = \mathcal{S}|_u$ is a shifted source

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When w is any finite sur-fix of u such that $w = u \cdot v$

the conditional probabilities $p_w/p_u = p_{(u.v)|u}$, denoted as $q_{v|u}$

are the fundamental probabilities of the source $S_{(u)}$.

A source is smooth if the probabilities $q_{i|w}$ satisfy

$$\exists p < 1, \qquad 0 < q_{i|w} \leq p, \qquad \forall (i, w) \in \Sigma \times \Sigma^\star$$

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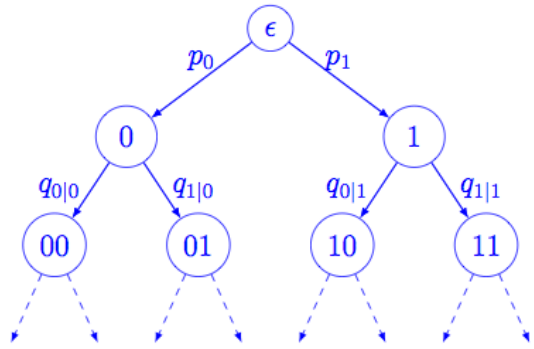

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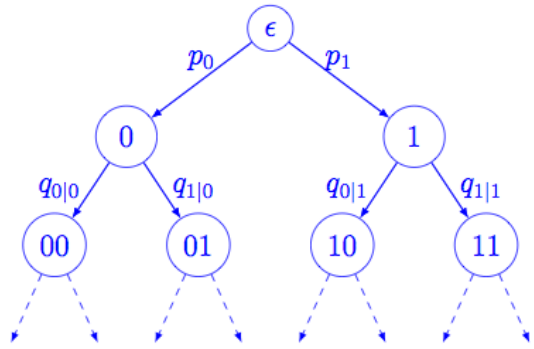

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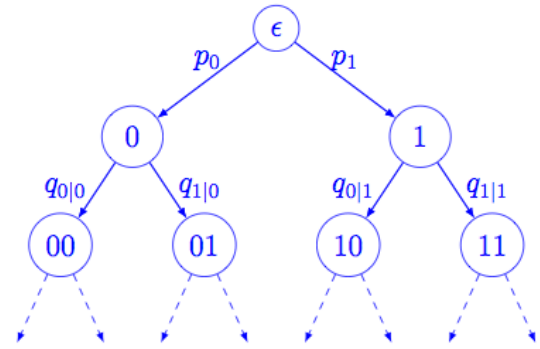

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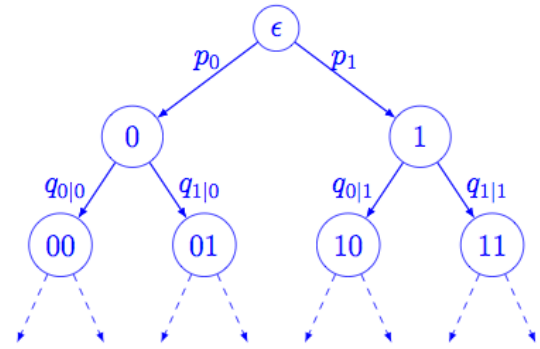

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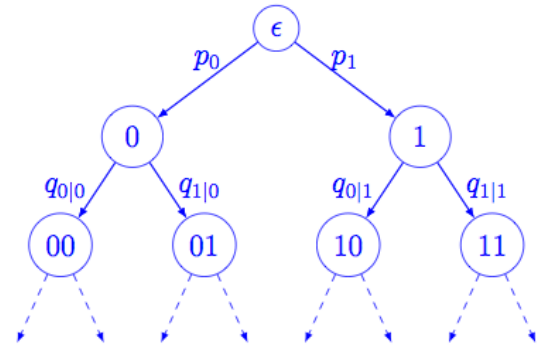

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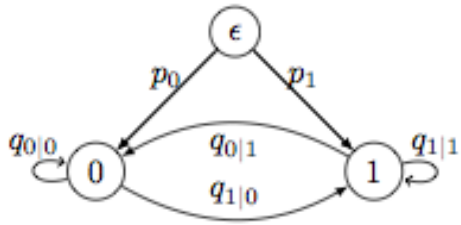

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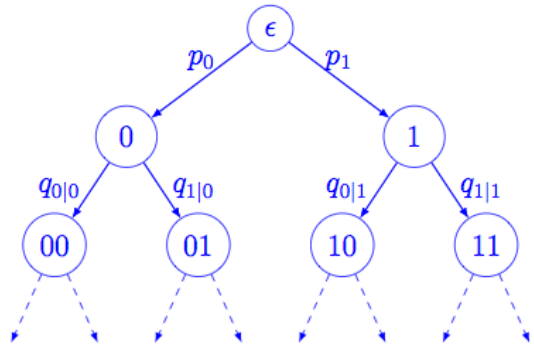

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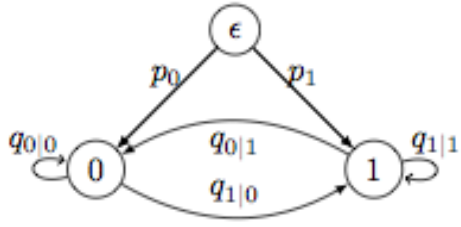

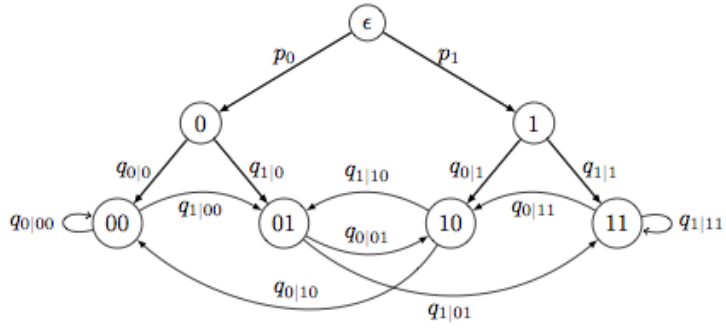

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For $\Re s$ close to 1, the dominant behavior of $\mathbf{Q}(s,u)$
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The analyses for tries only involve the first factor via $(I - u\mathbf{P}_s)^{-1}[1]$.

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is actually dictated by the first factor $(I - u\mathbf{P}_s)^{-1}$.
This explains the similarity of the results for dst and trie

However, the analysis for dst with a general source is more difficult:
– We need to study the operator $\mathbf{P}_s$ itself, not only $(I - u\mathbf{P}_s)^{-1}[1]$
– For a general source, the operator $\mathbf{P}_s$ is no longer a finite matrix, and the properties of $\mathbf{P}_s$ depend on the functional space on which it acts.

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Most of "good" dynamical sources are tame or even hyper-tame.

Our main results in the case of a general source – (I) Average -case analysis

Exact extension of previously known results for simple sources.

Consider a source $\mathcal{S}$ assumed to be a smooth, stationary, and tame, its basic operator $\mathbf{P}_s$, its dominant eigenvalue $\lambda(s)$.
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(c) The form of δ_n and R_n depend on $\mathbf{P}_s$ and $\lambda(s)$.

- δ_n exists when $\lambda(s)$ is periodic. It is a periodic function of $\log n$.
- $R_n \to 0$. It depends on the shape of the tameness region itself related to diophantine properties of probabilities.

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When the source is hyper-tame,

the speed of convergence towards the gaussian law is $O(\log n)^{-1/2}$.

More generally, it depends on the shape of the tameness region

(III) Some hints on the methods.

Fimal aim: an asymptotic estimate of $B_n(u) = n![z^n](e^z B(z,u))$

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– First, Step 1: Derivation of a system of basic equations
satisfied by a family of Poisson generating functions

– Then, Steps 2, 3, 4 : Using three main tools : Laplace; Rice, Mellin

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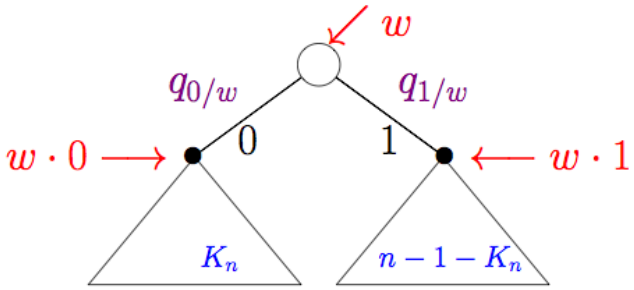

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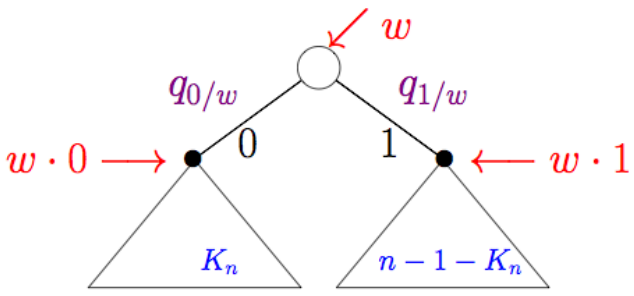

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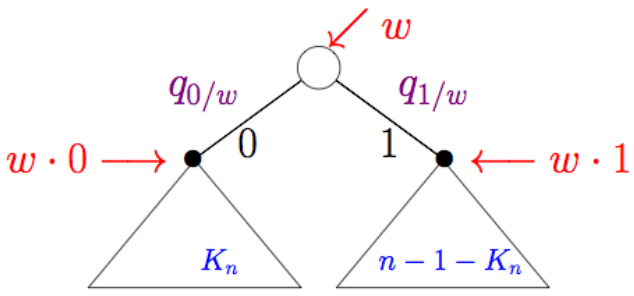

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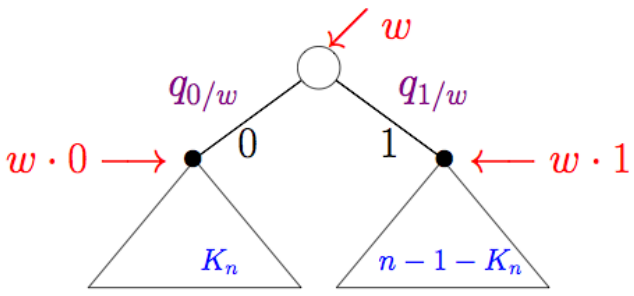

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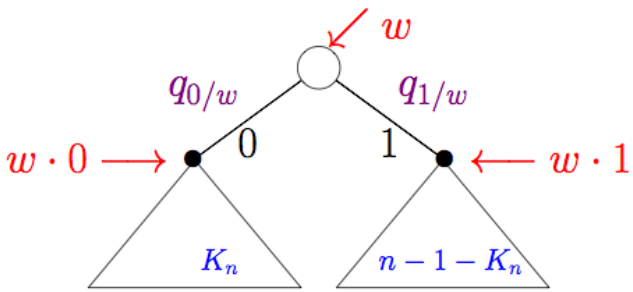

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which involves three main operations

– the derivation d/dz – the change of variables $z \mapsto qz$

– the shift on words $w \mapsto w.i$

In comparison, for tries, the derivation does not occur.

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Basic equations

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Needed: Nice properties of $s \mapsto (I - u\mathbf{P}_s)^{-1}$ on the left of $\Re s = 1$.

Many thanks for your attention