Compositions, canonical trees, acyclic digraphs and their common structural properties

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based on joint work with Clemens Heuberger and Daniel Krenn

Compositions

Definition

A *composition* of n is a representation of n as an ordered sum of positive integers: e.g.,

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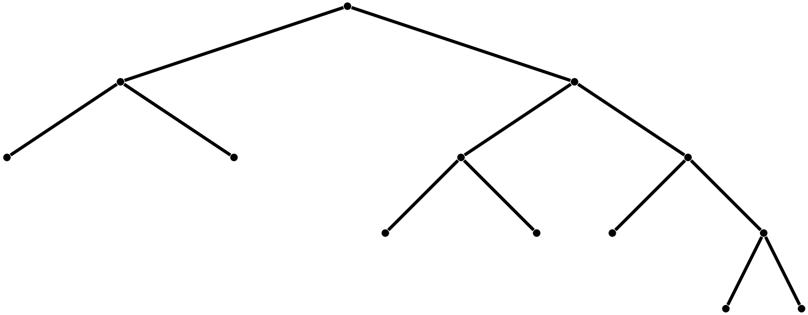
It is well known that there are 2^{n-1} compositions of n. The length is asymptotically normally distributed with mean $\frac{n+1}{2}$ and variance $\frac{n-1}{4}$, the largest summand is typically around $\log_2 n$, ...

Canonical trees

Definition

We call a rooted plane t-ary tree *canonical* if the vertex degrees are weakly increasing from left to right, as in the following example:



Canonical trees

Canonical t-ary trees are in bijection with *canonical compact prefix-free codes*: these are t-ary codes (codes over an alphabet of size t) such that:

- no word in the code is a proper prefix of another word (prefix-free),
- no word can be added to the code so that it remains prefix-free (compact),
- the lexicographic ordering corresponds to a non-decreasing ordering of word-lengths (canonical).

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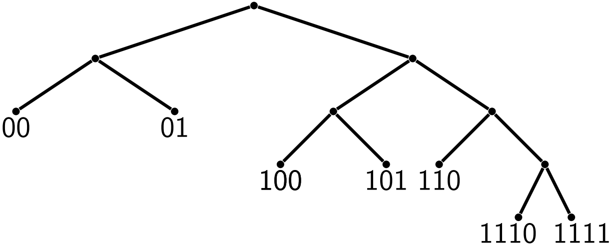
The following picture illustrates the bijection:



Canonical trees

Canonical trees are also in bijection with *partitions* of 1 into powers of t, i.e., representations of the form

$$1 = t^{-a_1} + t^{-a_2} + \cdots + t^{-a_n}$$

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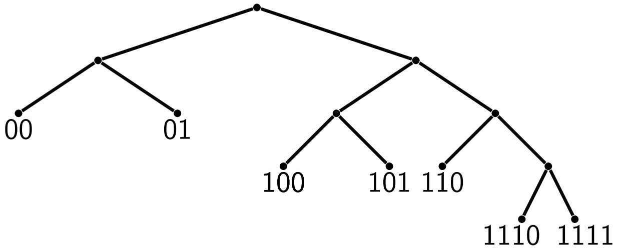
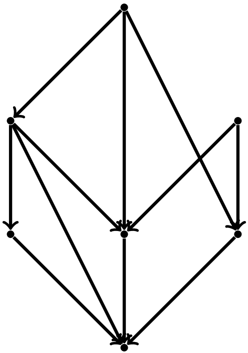


$$1 = 2^{-2} + 2^{-2} + 2^{-3} + 2^{-3} + 2^{-3} + 2^{-4} + 2^{-4}$$

Acyclic digraphs

A digraph is called *acyclic* if it does not contain a directed cycle.



An acyclic digraph is thus (in some sense) the directed analogue of a forest.

Generating functions and asymptotics

The generating function for canonical t-ary trees with a given number of internal vertices is

$$C(x) = \frac{\sum_{j \geq 0} (-1)^j x^{[j]} \prod_{i=1}^{j} \frac{x^{[i]}}{1-x^{[i]}}}{\sum_{j \geq 0} (-1)^j \prod_{i=1}^{j} \frac{x^{[i]}}{1-x^{[i]}}},$$

where $[j] = 1 + t + \cdots + t^{j-1}$.

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where $[j] = 1 + t + \cdots + t^{j-1}$. Singularity analysis yields an asymptotic formula:

Theorem (Boyd 1975, Komlos/Moser/Nemetz 1984, Flajolet/Prodinger 1987)

The number of canonical t-ary trees is asymptotically equal to $K_t \rho_t^n$ for some constants K_t and ρ_t. Moreover, $\rho_t \to 2$ as $t \to \infty$.

Generating functions and asymptotics

For acyclic digraphs, one needs a special type of generating function: if a_n is the number of labelled acyclic digraphs with n vertices, then

$$A(x) = \sum_{n=0}^{\infty} \frac{a_n}{n! 2^{\binom{n}{2}}} = \frac{1}{\sum_{n \geq 0} \frac{(-1)^n x^n}{n!} 2^{-\binom{n}{2}}}.$$

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Again, singularity analysis strikes:

Theorem (Robinson 1971)

The number of labelled acyclic digraphs of order n is

$$a_n \sim K n! \cdot 2^{\binom{n}{2}} z_0^{-n},$$

where $z_0 \approx 1.488079$.

A first connection

A canonical t-ary tree is uniquely determined if we know the number a_k of internal vertices at distance k from the root for all k. Clearly, $a_0 = 1$ (unless the root is the only vertex) and

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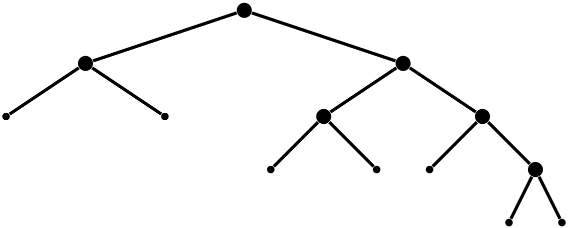

$$1 + 2 + 2 + 1 = 6$$

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There is a series of papers on properties of compositions with various local restrictions by Bender, Canfield and Gao (2005 –).

Transfer matrices

Let $C_k(x)$ be the generating function for canonical t-ary trees with the property that there are exactly k vertices with maximum distance from the root. Then

$$C_k(x) = x^k \sum_{j=\lceil \frac{k}{t} \rceil}^{\infty} C_j(x) + [k = 1]x.$$

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This is a linear system of equations with an *infinite* transfer matrix. For $t = 2$, the matrix looks like this:

$$\begin{bmatrix} x & x & x & x & x & \cdots \\ x^2 & x^2 & x^2 & x^2 & x^2 & \cdots \\ 0 & x^3 & x^3 & x^3 & x^3 & \cdots \\ 0 & x^4 & x^4 & x^4 & x^4 & \cdots \\ 0 & 0 & x^5 & x^5 & x^5 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Transfer matrices

A similar idea works for acyclic digraphs: every such digraph has a number of *sinks* (vertices of outdegree 0). For each vertex v, let the *level* be the length of the longest directed path starting at v (so that the level of a sink is 0). If $a_{n,k}$ is the number of labelled acyclic digraphs of order n with k vertices at the highest level, then we have the recursion

$$a_{n,k} = \binom{n}{k} \sum_{j=1}^{\infty} a_{n-k,j} 2^{k(n-k)} (1 - 2^{-j})^k$$

with $a_{k,k} = 1$ for $k \geq 1$ and $a_{n,k} = 0$ for $n < k$.

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This can again be seen as a system of linear equations with an infinite transfer matrix.

How to proceed from there

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The similarities in the functional equations explain why shape parameters of canonical trees and acyclic digraphs behave in a similar way (which is quite different from other random tree models such as Galton-Watson trees or recursive trees).

Some structural results: height

Theorem (McKay 1989)

The height (longest directed path) of a random labelled (or unlabelled) acyclic digraph has a Gaussian limiting distribution with linear mean and variance.

Theorem (Heuberger/Krenn/SW 2013)

The height of a random canonical t-ary tree has a Gaussian limiting distribution with linear mean and variance.

Some structural results: width

Theorem (Krenn/SW 2013+)

The width (largest number of vertices on a single level) of a random acyclic digraph of order n is concentrated around $C_1\sqrt{\log n}$.

Theorem (Heuberger/Krenn/SW 2013)

The width of a random canonical t-ary tree of order n is concentrated around $C_2 \log n$.

Some structural results: profile

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- For fixed ℓ, the number of vertices at level ℓ asymptotically follows a discrete limiting distribution that depends on ℓ.
- If $\ell = H - m$ for fixed m, where H denotes the height, then the number of vertices at level ℓ asymptotically follows a discrete limiting distribution that depends on m.
- If $1 \ll \ell \ll H$, then the number of vertices at level ℓ asymptotically follows a discrete limiting distribution that does not depend on ℓ.